Positive solutions for superdiffusive mixed problems

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We study a semilinear parametric elliptic equation with superdiffusive reaction and mixed boundary conditions. Using variational methods, together with suitable truncation techniques, we prove a bifurcation-type theorem describing the nonexistence, existence and multiplicity of positive solutions.

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1. Introduction

Let \(Ω \subseteq \mathbb{R}^N\) be a bounded domain with a \(C^2\)-boundary \(\partial Ω\) and let \(Σ_1, Σ_2 \subseteq \partial Ω\) be two \((N−1)\)-dimensional \(C^2\)-submanifolds of \(\partial Ω\) such that \(\partial Ω = Σ_1 \cup Σ_2\), \(Σ_1 \cap Σ_2 = \emptyset\), \(|Σ_1|_{N−1} \in (0, |∂Ω|_{N−1})\), and \(Σ_1 \cap Σ_2 = Γ\). Here, \(|·|_{N−1}\) denotes the \((N−1)\)-dimensional Hausdorff (surface) measure and \(Γ \subset ∂Ω\) is a \((N−2)\)-dimensional \(C^2\)-submanifold of \(∂Ω\).

In this paper, we study the following logistic-type elliptic problem:

\[
\begin{cases}
-Δu(z) = λu(z)^{q−1} - f(z, u(z)) & \text{in } Ω, \\
u|_{Σ_1} = 0, & ∂u|_{Σ_2} = 0, & u > 0, & λ > 0.
\end{cases}
\]

When \(f(z, x) = x^{r−1}\) with \(r \in (2, 2^*)\), we get the classical logistic equation, which is important in biological models (see Gurtin & Mac Camy [1]). Depending on the value of \(q > 1\), we distinguish three cases:

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(i) \(1 < q < 2\) (subdiffusive logistic equation); (ii) \(2 = q < r\) (equidiffusive logistic equation); (iii) \(2 < q < r\) (superdiffusive logistic equation). In this paper, we deal with the third situation (superdiffusive case), which exhibits bifurcation-type phenomena for large values of the parameter \(\lambda > 0\) (see also [2]).

Let \(E_{\Sigma_1} = \{ u \in H^1(\Omega) : u|_{\Sigma_1} = 0 \}\). This space is defined as the closure of \(C^1_c(\Omega \cup \Sigma_1)\) with respect to the \(H^1(\Omega)\)-norm. Since \(|\Sigma_1| \cdot \|N\|_{L^1} > 0\), we know that for the space \(E_{\Sigma_1}\), the Poincaré inequality holds (see Gasinski & Papageorgiou [3, Problem 1.139, p. 58]). So, \(E_{\Sigma_1}\) is a Hilbert space equipped with the norm \(\|u\| = \|Du\|_2\). Let \(A \in L(E_{\Sigma_1}, E_{\Sigma_1}^\ast)\) be defined by \((A(u), h) = \int_\Omega (Du, Dh)_{\mathbb{R}^N} \, dz\) for all \(u, h \in E_{\Sigma_1}\). We denote by \(N_f\) the Nemitsky map associated with \(f\), that is, \(N_f(u)(\cdot) = f(\cdot, u(\cdot))\) for all \(u \in E_{\Sigma_1}\).

The hypotheses on the perturbation term \(f(z, x)\) are the following:

\[ H(f) : f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function such that for almost all } z \in \Omega, f(z, 0) = 0, f(z, x) \geq 0 \text{ for all } x > 0, \text{ and} \]

(i) \(f(z, x) \leq a(z)(1 + x^{q-1})\) for almost all \(z \in \Omega\) and all \(x \geq 0\), with \(a \in L^\infty(\Omega), 2 < q < r < 2^*\);

(ii) \(\lim_{x \to +\infty} \frac{f(z, x)}{x^q} = +\infty\) uniformly for almost all \(z \in \Omega\), and the mapping \(x \mapsto \frac{f(z, x)}{x}\) is nondecreasing on \((0, +\infty)\) for almost all \(z \in \Omega\);

(iii) \(0 \leq \liminf_{x \to 0^+} \frac{f(z, x)}{x} \leq \limsup_{x \to 0^+} \frac{f(z, x)}{x} \leq \tilde{\eta}\) uniformly for almost all \(z \in \Omega\);

(iv) for every \(\rho > 0\), there exists \(\tilde{\xi}_\rho > 0\) such that for almost all \(z \in \Omega\) the function \(x \mapsto \tilde{\xi}_\rho x - f(z, x)\) is nondecreasing on \([0, \rho]\).

The following functions satisfy hypotheses \(H(f)\): (i) \(f(x) = x^{q-1}\) for all \(x \geq 0\) with \(2 < q < r < 2^*\); (ii) \(f(x) = x^{q-1}\left[\ln(1 + x) + \frac{1}{q} \frac{1}{1 + x}\right]\) for all \(x \geq 0\), with \(2 < q < 2^*\).

Let \(\mathcal{L} = \{ \lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution} \} \) and let \(S(\lambda)\) denote the set of positive solutions of problem \((P_\lambda)\). Let \(\lambda_* = \inf \mathcal{L}\) (if \(\mathcal{L} = \emptyset\), then \(\inf \mathcal{L} = +\infty\)).

By a solution of problem \((P_\lambda)\), we understand a function \(u \in E_{\Sigma_1}\) such that \(u \geq 0, u \neq 0\) and \((A(u), u) = \int_\Omega [\lambda u^{q-1} - f(z, u)] \, dz\) for all \(u \in E_{\Sigma_1}\).

We refer to Bonanno, D’Agui & Papageorgiou [4], Filippucci, Pucci & Rădulescu [5], and Li, Ruf, Guo & Niu [6] for related results. We also refer to the monograph by Pucci & Serrin [7] for more results concerning the abstract setting of this paper.

2. A bifurcation-type theorem

**Proposition 1.** If hypotheses \(H(f)\) hold, then \(S(\lambda) \subseteq C^{1, \alpha}(\Omega) \cap C^0,\alpha(\overline{\Omega})\) with \(\alpha \in (0, 1/2)\). For all \(u \in S(\lambda)\) we have \(u(z) > 0\) for all \(z \in \Omega\) and \(\lambda_* > 0\).

**Proof.** From DiBenedetto [8] and Colorado & Peral [9], we know that if \(u \in S(\lambda)\) then \(u \in C^{1, \alpha}(\Omega) \cap C^0,\alpha(\overline{\Omega})\) with \(\alpha \in (0, 1/2)\). Moreover, using Harnack’s inequality, we deduce that if \(u \in S(\lambda)\) then \(u(z) > 0\) for all \(z \in \Omega\). Let \(\tilde{\lambda}_1\) be the smallest eigenvalue of \(-\Delta\) with mixed boundary conditions. From Colorado & Peral [9, p. 482], we know that \(\tilde{\lambda}_1 = \inf \left\{ \frac{\|Du\|^2}{\|u\|^2} : u \in E_{\Sigma_1} \setminus \{0\} \right\} > 0\). By \(H(f)\)(i), (iii), there exists \(\lambda_0 > 0\) such that

\[ \lambda_0 x^{q-1} - f(z, x) \leq \tilde{\lambda}_1 x \text{ for almost all } z \in \Omega, \text{ and all } x \geq 0 \quad (1) \]

(recall that \(2 < q < r\)). Let \(\lambda \in (0, \lambda_0)\) and suppose that \(\lambda \in \mathcal{L}\). Then there exists \(u_\lambda \in S(\lambda)\) and by using Green’s identity, we get

\[ A(u_\lambda) = \lambda u_\lambda^{q-1} - N_f(u_\lambda) \text{ in } E_{\Sigma_1}^\ast. \quad (2) \]

We act on \((2)\) with \(u_\lambda \in E_{\Sigma_1}\) and obtain \(\|Du_\lambda\|^2 = \lambda \|u_\lambda\|^q - \int_\Omega f(z, u_\lambda) u_\lambda dz < \tilde{\lambda}_1 \|u_\lambda\|^2\) (see \((1)\) and recall that \(\lambda < \lambda_0, u_\lambda(z) > 0\) for all \(z \in \Omega\), which contradicts the definition of \(\tilde{\lambda}_1\). Therefore \(\lambda \not\in \mathcal{L}\) and we have \(0 < \lambda_0 \leq \lambda_* = \inf \mathcal{L}\). □
Proposition 2. If hypotheses $H(f)$ hold, then $\mathcal{L} \neq \emptyset$ and “$\lambda \in \mathcal{L}, \eta > \lambda \Rightarrow \eta \in \mathcal{L}$”.

Proof. Fix $\lambda > 0$ and let $\varphi_\lambda : E_{\Sigma_1} \to \mathbb{R}$, $\varphi_\lambda(u) = \frac{1}{2} \| Du \|^2_2 - \frac{\lambda}{q} \| u^+ \|^q_q + \int_{\Omega} F(z, u)dz$, where $F(z, x) = \int_0^x f(z, s)ds$. Then $\varphi_\lambda \in C^1(E_{\Sigma_1})$ and $\varphi_\lambda$ is sequentially weakly lower semicontinuous. Hypotheses $H(f)(i), (ii)$ imply that given $\xi > 0$, we can find $c_1 = c_1(\xi) > 0$ such that $F(z, x) \geq \frac{\xi}{q} x^q - c_1$ for almost all $z \in \Omega$ and for all $x \geq 0$. Thus, for all $u \in E_{\Sigma_1}$ we have $\varphi_\lambda(u) \geq \frac{1}{2} \| Du \|^2_2 + \frac{\xi}{q} \| u^+ \|^q_q - c_1 |\Omega|$. Choosing $\xi > \lambda$, we deduce that $\varphi_\lambda$ is coercive. So, by the Weierstrass–Tonelli theorem, there exists $u_\lambda \in E_{\Sigma_1}$ such that

$$\varphi_\lambda(u_\lambda) = \inf \{ \varphi_\lambda(u) : u \in E_{\Sigma_1} \} = m_\lambda. \tag{3}$$

Fix $\bar{u} \in E_{\Sigma_1} \cap \bar{C(\Omega)}$ with $u(z) > 0$ for all $z \in \Omega$. For large enough $\lambda > 0$ we have $\varphi_\lambda(\bar{u}) < 0$, hence $\varphi_\lambda(u_\lambda) = m_\lambda < 0 = \varphi_\lambda(0)$ (see (3)). Thus, $u_\lambda \neq 0$. By (3), $\varphi_\lambda'(u_\lambda) = 0$, hence

$$A(u_\lambda) = \lambda (u_\lambda^+)^q - N_f(u_\lambda) \in E^*_\Sigma_1. \tag{4}$$

We act on (4) with $-u_\lambda^+ \in E_{\Sigma_1}$ and obtain $\| Du_\lambda \|^2_2 = 0$, hence $u_\lambda \geq 0$. So, relation (4) becomes $A(u_\lambda) = \lambda u_\lambda^q - N_f(u_\lambda)$. By Green’s identity, $u_\lambda \in S(\lambda)$, hence $\lambda \in \mathcal{L} \neq \emptyset$.

Next, let $\lambda \in \mathcal{L}$ and $\eta > \lambda$. Choose $\vartheta \in (0, 1)$ such that $\lambda = \vartheta^q - \vartheta \eta$ (recall that $2 < q$). Also, let $u_\lambda \in S(\lambda) \subseteq C^{1,\alpha}(\Omega) \cap C^0(\mathbb{R})$ with $\alpha \in (0, 1/2)$. Let $u = \vartheta u_\lambda$. Then

$$A(u) = \vartheta A(u_\lambda) = \vartheta \left[ \lambda u_\lambda^q - N_f(u_\lambda) \right] \in E^*_\Sigma_1. \tag{5}$$

From hypothesis $H(f)(ii)$ and since $u_\lambda(z), u(z) > 0$ for all $z \in \Omega$, we have for a.a. $z \in \Omega$

$$\frac{f(z, u(z))}{u(z)} \leq \frac{f(z, u_\lambda(z))}{u_\lambda(z)} \Rightarrow f(z, u(z)) \leq \vartheta f(z, u_\lambda(z)) \text{ (recall that } u = \vartheta u_\lambda). \tag{6}$$

Using (5) in (6) and since $\vartheta \in (0, 1)$, we obtain

$$A(u) \leq \vartheta^{q-1} \eta u_\lambda^q - N_f(u) \leq \eta u_\lambda^{q-1} - N_f(u) \in E^*_\Sigma_1. \tag{7}$$

We introduce the following Carathéodory truncation of the reaction term in problem $(P_\eta)$

$$g_\eta(z, x) = \begin{cases} \eta u(z)^{q-1} - f(z, u(z)) & \text{if } x \leq u(z) \\ \eta x^{q-1} - f(z, x) & \text{if } u(z) < x. \end{cases} \tag{8}$$

Let $G_\eta(z, x) = \int_0^x g_\eta(z, s)ds$ and define $\varphi_\eta : E_{\Sigma_1} \to \mathbb{R}$ by $\varphi_\eta(u) = \frac{1}{2} \| Du \|^2_2 - \int_{\Omega} G_\eta(z, u)dz$.

Hypotheses $H(f)(i), (ii)$ imply that given $\xi > 0$, we can find $c_2 = c_2(\xi) > 0$ such that

$$\eta x^{q-1} - f(z, x) \leq (\eta - \xi) x^{q-1} + c_2 \text{ for almost all } z \in \Omega \text{ and all } x \geq 0. \tag{9}$$

Then for all $u \in E_{\Sigma_1}$, we have

$$\varphi_\eta(u) \geq \frac{1}{2} \| Du \|^2_2 + \frac{\xi - \eta}{q} \| u^+ \|^q_q - c_3 \text{ for some } c_3 > 0 \text{ (see (8), (9)).} \tag{10}$$

Choosing $\xi > \eta$, we see from (10) that $\varphi_\eta$ is coercive. This function is also sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, there exists $u_\eta \in E_{\Sigma_1}$ such that $\varphi_\eta(u_\eta) = \inf \{ \varphi_\eta(u) : u \in E_{\Sigma_1} \}$, hence $\varphi_\eta'(u_\eta) = 0$. We deduce that

$$A(u_\eta) = N_{g_\eta}(u_\eta) \in E^*_\Sigma_1. \tag{11}$$
We act on (11) with \((u - u_\eta)^+ \in E_\Sigma\). By (8) and (7) we have
\[
\langle A(u_\eta), (u - u_\eta)^+ \rangle = \int_\Omega [\eta u_\eta^{q-1} - f(z, u)](u - u_\eta)^+ dz \geq \langle A(u), (u - u_\eta)^+ \rangle
\]
and
\[
\Rightarrow \langle A(u - u_\eta), (u - u_\eta)^+ \rangle \leq 0 \Rightarrow \|D(u - u_\eta)^+\|^2_2 \leq 0 \Rightarrow u \leq u_\eta.
\]
(12)

Using (8) and (12) we see that relation (11) becomes \(A(u_\lambda) = \eta u_\eta^{q-1} - N_f(u_\eta)\) in \(E_\Sigma^*\). Thus, by Proposition 1, we have \(u_\eta \in S(\eta) \subseteq C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})\). Therefore \(\eta \in \mathcal{L}\). We also observe that Proposition 2 implies \((\lambda_\ast, +\infty) \subseteq \mathcal{L}\).

**Proposition 3.** If hypotheses \(H(f)\) hold and \(\lambda > \lambda_\ast\), then problem \((P_\lambda)\) has at least two positive solutions \(u_0, \hat{u} \in E_\Sigma \cap C^{0,\alpha}(\overline{\Omega})\) for \(\alpha \in (0, 1/2)\) with \(0 < u_0(z), \hat{u}(z)\) for all \(z \in \Omega\).

**Proof.** Let \(\mu \in (\lambda_\ast, \lambda)\). By Proposition 2 we know that \(\mu \in \mathcal{L}\). Hence we can find \(u_\mu \in S(\mu) \subseteq E_\Sigma \cap C^{0,\alpha}(\overline{\Omega})\) with \(\alpha \in (0, 1/2), u_\mu(z) > 0\) for all \(z \in \Omega\). We have \(A(u_\mu) = \mu u_\mu^{q-1} - N_f(u_\mu)\) in \(E_\Sigma^*\). Next, we define the following Carathéodory function
\[
\hat{h}_\lambda(z, x) = \begin{cases} 
\lambda u_\mu(z)^{q-1} - f(z, u_\mu(z)) & \text{if } x \leq u_\mu(z) \\
\lambda x^{q-1} - f(z, x) & \text{if } u_\mu(z) < x.
\end{cases}
\]
(13)

Let \(\hat{H}_\lambda(z, x) = \int_0^x \hat{h}_\lambda(z, s)ds\) and let \(\hat{\psi}_\lambda : E_\Sigma \to \mathbb{R}, \hat{\psi}_\lambda(u) = \frac{1}{2}\|Du\|^2_2 - \int_\Omega \hat{H}_\lambda(z, u)dz\). Then \(\hat{\psi}_\lambda\) is coercive and sequentially weakly lower semicontinuous. Thus, we can find \(u_\theta \in E_\Sigma\), such that \(\hat{\psi}_\lambda(u_\theta) = \inf\{\hat{\psi}_\lambda(u) : u \in E_\Sigma\}\), and hence \(\hat{\psi}_\lambda'(u_\theta) = 0\). Thus, \(A(u_\theta) = \lambda_{1\lambda}(u_\theta)\). Using (13) and reasoning as in the proof of Proposition 2 we deduce that \(u_\mu \leq u_\theta\). By Colorado & Peral [9, Theorem 6.6], we have \(u_\theta \in E_\Sigma \cap C^{0,\alpha}(\overline{\Omega})\) with \(\alpha \in (0, 1/2)\) and \(u_\theta > 0\) in \(\Omega\) (by Harnack’s inequality).

Let \(\rho_0 = \|u_\theta\|\) and let \(\hat{\xi}_{\rho_0} > 0\) be as postulated in hypothesis \(H(f)\)(iv). We have
\[
\begin{align*}
-\Delta u_\theta(z) + \hat{\xi}_{\rho_0} u_\theta(z) = & \lambda u_\theta(z)^{q-1} - f(z, u_\theta(z)) + \hat{\xi}_{\rho_0} u_\theta(z) \quad \text{in } \Omega, \\
u_\theta|_{\Sigma_1} = & 0, \quad \frac{\partial u_\theta}{\partial n}|_{\Sigma_2} = 0
\end{align*}
\]
(14)

and
\[
\begin{align*}
-\Delta u_\mu(z) + \hat{\xi}_{\rho_0} u_\mu(z) = & \mu u_\mu(z)^{q-1} - f(z, u_\mu(z)) + \hat{\xi}_{\rho_0} u_\mu(z) \quad \text{in } \Omega, \\
\hat{u}_\mu|_{\Sigma_1} = & 0, \quad \frac{\partial u_\mu}{\partial n}|_{\Sigma_2} = 0
\end{align*}
\]
(15)

Let \(\hat{y} = u_\theta - u_\mu \geq 0\). Since \(\lambda > \mu, u_\theta \geq u_\mu\), from (14), (15), and \(H(f)\)(iv) we have
\[
-\Delta \hat{y}(z) + \hat{\xi}_{\rho_0} \hat{y}(z) = \lambda u_\theta(z)^{q-1} - \mu u_\mu(z)^{q-1} + [\hat{\xi}_{\rho_0} u_\theta(z) - f(z, u_\theta(z)) - -[\hat{\xi}_{\rho_0} u_\mu(z) - f(z, u_\mu(z))] \geq 0 \quad \text{in } \Omega.
\]

Let \(v_1 \in E_\Sigma\) be the unique function satisfying \(-\Delta v(z) + \hat{\xi}_{\rho_0} v(z) = 1 \quad \text{in } \Omega, v|_{\Sigma_1} = 0\), and \(\frac{\partial v}{\partial n}|_{\Sigma_2} = 0\). Then \(v_1 \in C^{1,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega})\) with \(\alpha \in (0, 1/2)\) (see [8,9]) and \(v_1 > 0\) in \(\Omega\). By Lemma 2.1 of Barletta, Livrea & Papagegiou [10] (see also Lemma 5.3 of Colorado & Peral [9]), we can find \(\theta > 0\) such that
\[
\partial v_1(z) \leq u_\mu(z) \quad \text{and} \quad \partial v_1(z) \leq \hat{y}(z) \Rightarrow \partial v_1(z) \leq u_\theta(z) \leq u_0(z) - \partial v_1(z) \quad \text{for all } z \in \overline{\Omega}.
\]
(16)

Let \(\hat{C}_1 = \{y \in E_\Sigma \cap C^{0,\alpha}(\overline{\Omega}) : \frac{\|y\|}{v_1} < \infty\}\) and \([u_\mu] = \{u \in E_\Sigma : u_\mu(z) \leq u(z), \text{a.a. } z \in \Omega\}\). We claim that if \(\tilde{B}_1(0) = \{y \in \hat{C}_1 : \frac{\|y\|}{v_1} \leq 1\}\), then \(u_0 - \partial \tilde{B}_1(0) \subseteq [u_\mu] \cap \hat{C}_1\). To see this, let \(y \in \tilde{B}_1(0)\). Then
\[
-\partial v_1(z) \leq y(z) \leq v_1(z) \quad \text{for all } z \in \overline{\Omega}.
\]
(17)
Fix $z \in \overline{\Omega}$. If $g(z) > 0$, then $0 \leq u_{\mu}(z) \leq u_{\mu}(z) + \vartheta y(z) \leq u_{\mu}(z) + \vartheta v_1(z) \leq u_0(z)$ (see (16), (17)), hence $u_{\mu}(z) \leq u_0(z) - \vartheta y(z)$. If $g(z) < 0$, then $0 \leq u_{\mu}(z) - \vartheta v_1(z) \leq u_{\mu}(z) + \vartheta y(z) \leq u_{\mu}(z) + \vartheta v_1(z) \leq u_0(z)$ (see (16), (17)), hence $u_{\mu}(z) \leq u_0(z) - \vartheta y(z)$. We conclude that $u_{\mu} \in u_0 - \vartheta \hat{B}_1(0)$, which proves the claim. It follows that

$$u_0 \in \text{int}_{\hat{C}_1} [u_{\mu}] \cap C(\overline{\Omega}).$$

(18)

By (13) it is clear that

$$\hat{\psi}_\lambda(u) = \varphi_\lambda(u) + c_4 \text{ for some } c_4 \in \mathbb{R} \text{ and for all } u \in [u_{\mu}).$$

(19)

It follows from (18) and (19) that $u_0$ is a local $\hat{C}_1$-minimizer of $\varphi_\lambda$.

**Claim.** $u_0$ is a local $E_{\Sigma_1}$-minimizer of $\varphi_\lambda$.

Suppose that this assertion is not true. Then for every $\rho > 0$, we have $\inf\{\varphi_\lambda(u_0 + y) : y \in E_{\Sigma_1}, ||y|| \leq \rho \} < \varphi_\lambda(u_0)$. By the Weierstrass–Tonelli theorem, there exists $y_\rho \in E_{\Sigma_1} \setminus \{0\}$, $||y_\rho|| \leq \rho$ such that $\varphi_\lambda(u_0 + y_\rho) = \inf\{\varphi_\lambda(u_0 + y) : y \in E_{\Sigma_1}, ||y|| \leq \rho \} < \varphi_\lambda(u_0)$. By the Lagrange multiplier rule, there exists $\vartheta \leq 0$ such that $(1 - \vartheta) (A(u_\rho), h) = \lambda \int_{\Omega} (u_\rho^+)^{q-1} h dz - \int_{\Omega} f(z, u_\rho) h dz$ for all $h \in E_{\Sigma_1}$, with $u_\rho = u_0 + y_\rho \in E_{\Sigma_1}$. It follows that $\Delta u_\rho(z) = \frac{1}{1 - \vartheta} [\lambda u_\rho^+(z)^{q-1} - f(z, u_\rho(z))]$ in $\Omega$, hence

$$-\Delta u_\rho(z) + \hat{\xi}_{\rho_0} u_\rho(z) = \frac{1}{1 - \vartheta} [\lambda u_\rho^+(z)^{q-1} + f(z, u_\rho(z))] + \hat{\xi}_{\rho_0} u_\rho(z) \text{ in } \Omega,$$

(20)

with $\hat{\xi}_{\rho_0} > 0$ as before resulting from hypothesis $H(f)$ (recall that $\rho_0 = ||u_0||_\infty$). Also,

$$-\Delta u_0(z) + \hat{\xi}_{\rho_0} u_0(z) = \lambda u_0(z)^{q-1} - f(z, u_0(z)) + \hat{\xi}_{\rho_0} u_0(z) \text{ in } \Omega.$$

(21)

From (20) and (21) we obtain

$$-\Delta y_\rho(z) + \hat{\xi}_{\rho_0} y_\rho(z) = g_\lambda^\varrho(z) \text{ in } \Omega,$$

(22)

with $g_\lambda^\varrho(z) = \frac{1}{1 - \vartheta} [\lambda u_\rho^+(z)^{q-1} - f(z, u_\rho(z))] - \lambda u_0(z)^{q-1} + f(z, u_0(z)) + \hat{\xi}_{\rho_0} y_\rho(z)$. By (22) and Colorado & Peral [9], there exist $c_5 > 0$ and $\alpha \in (0, 1/2)$ such that

$$y_\rho \in C^{0, \alpha}(\overline{\Omega}) \text{ and } ||y_\rho||_{C^{0, \alpha}(\overline{\Omega})} \leq c_5 \text{ for all } \rho \in (0, 1].$$

(23)

Exploiting the compact embedding of $C^{0, \alpha}(\overline{\Omega})$ into $C(\overline{\Omega})$, we have $y_\rho \to 0$ in $C(\overline{\Omega})$ as $\rho \to 0^+$. Thus, by the definition of $g_\lambda^\varrho$, there exists $\tau^*_{\rho} > 0$ such that

$$||g_\lambda^\varrho||_\infty \leq \tau^*_{\rho} \text{ for all } \rho \in (0, 1] \text{ and } \tau^*_{\rho} \to 0^+ \text{ as } \rho \to 0^+.$$

(24)

Let $y_\rho = \frac{1}{\tau^*_{\rho}} y_\rho$. Then by (24) $-\Delta (y_\rho - v_1(z) + \hat{\xi}_{\rho_0}(y_\rho - v_1)(z) = \frac{1}{\tau^*_{\rho}} g_\lambda^\varrho(z) - 1 \leq 0$. We deduce that $||D(y_\rho - v_1) + \hat{\xi}_{\rho_0}(y_\rho - v_1)||^2 \leq 0$, hence $y_\rho \leq \tau^*_{\rho} v_1$.

Also, we have $-\Delta (-y_\rho - v_1(z) + \hat{\xi}_{\rho_0}(-y_\rho - v_1)(z)) = -\frac{1}{\tau^*_{\rho}} g_\lambda^\varrho(z) - 1 \leq 0$ in $\Omega$ and so as above we obtain that $-\tau^*_{\rho} v_1 \leq y_\rho$. Therefore we have proved that $-\tau^*_{\rho} v_1 \leq y_\rho \leq \tau^*_{\rho} v_1$. These relations show that $y_\rho \in \hat{C}_1$ and $||\frac{y_\rho}{v_1}||_\infty \leq \tau^*_{\rho}$ for all $\rho \in (0, 1]$, hence $y_\rho \to 0$ in $\hat{C}_1$ as $\rho \to 0^+$. Therefore for small $\rho \in (0, 1]$ we have $\varphi_\lambda(u_0 + y_\rho) < \varphi_\lambda(u_0)$, which contradicts the fact that $u_0$ is a local $\hat{C}_1$-minimizer of $\varphi_\lambda$. This proves the claim.

Since $f \geq 0$, for all $u \in E_{\Sigma_1}$ we have $\varphi_\lambda(u) \geq \frac{1}{2} ||Du||_2^2 - \frac{1}{q} ||u||_q^q \geq \frac{1}{2} ||Du||_2^2 - c_6 ||Du||_2^q$ for some $c_6 > 0$. Since $q > 2$, we deduce that $u = 0$ is a local minimizer of $\varphi_\lambda$. We assume that the set of critical points of $\varphi_\lambda$
is finite (otherwise we already have an infinity of positive solutions for \((P_\lambda)\) for \(\lambda > \lambda_s\) and so we are done) and that \(\varphi_\lambda(0) \leq \varphi_\lambda(u_0)\) (the reasoning is similar if the opposite inequality holds). The claim implies that we can find small enough \(\rho \in (0, \|u_0\|)\) such that \(0 = \varphi_\lambda(0) \leq \varphi_\lambda(u) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\lambda^0\). Thus, we can apply the mountain pass theorem. So, there exists \(\hat{u}\) that

\[\phi \in \mathcal{P}\]

from hypotheses \(\text{Proposition 4.}\)

**Proof.** Let \(\{\lambda_n\}_{n \geq 1} \subseteq (\lambda_s, +\infty)\) be such that \(\lambda_n \downarrow \lambda_s\). We find \(u_n \in S(\lambda_n)\) such that

\[A(u_n) = \lambda_n u_n^{\frac{q-1}{q}} - N_f(u_n)\]

for all \(n \in \mathbb{N}\). (25)

Hypotheses \(H(f)(i), (ii)\) imply that given any \(\xi > 0\), we find \(c_7 = c_7(\xi) > 0\) such that

\[f(z, x) \geq \xi x^{q-1} - c_7\]

for almost all \(z \in \Omega\) and all \(x > 0\). (26)

We act on (25) with \(u_n \in E_{\Sigma_1}\) and then use (26). We obtain \(\|Du_n\|^2 \leq (\lambda_n - \xi) \|u_n\|^q + c_7|\Omega|N\). Choosing \(\xi > \lambda_1 \geq \lambda_n\) for all \(n \in \mathbb{N}\), we have \(\|Du_n\|^2 \leq c_7|\Omega|N\) for all \(n \in \mathbb{N}\), hence \(\{u_n\}_{n \geq 1} \subseteq E_{\Sigma_1}\) is bounded. By passing to a subsequence if necessary, we may assume that

\[u_n \rightharpoonup u * \text{ in } E_{\Sigma_1} \text{ and } u_n \rightarrow u \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.\] (27)

In (25) we pass to the limit as \(n \rightarrow \infty\) and use (27). Then \(A(u_*) = \lambda_s u_*^{q-1} - N_f(u_*)\). Thus, \(u_* \in E_{\Sigma_1}\) and \(u_* \geq 0\) is a solution of \((P_{\lambda_s})\). We also notice that \(\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u_* \rangle = 0\), hence \(\|Du_n\| \rightarrow \|Du_*\|\). Using the Kadec–Klee property we deduce that \(u_n \rightarrow u_* \text{ in } E_{\Sigma_1}\).

**Claim.** \(u_* \neq 0\).

Arguing by contradiction, suppose that \(u_* = 0\). Then \(\|u_n\| \rightarrow 0\). Let \(y_n = \frac{u_n}{\|u_n\|}, \ n \in \mathbb{N}\). Then \(\|y_n\| = 1, \ y_n \geq 0\) for all \(n \in \mathbb{N}\). From (25) we have

\[A(y_n) = \lambda_n y_n^{\frac{q-1}{q}} - \frac{N_f(u_n)}{\|u_n\|} \text{ for all } n \in \mathbb{N}.\] (28)

From hypotheses \(H(f)(i), (iii)\), we see that we can find \(\eta > \hat{\eta}\) and \(c_8 > 0\) such that

\[f(z, x) \leq \eta x + c_8 x^{r-1}\] for a.a. \(z \in \Omega, \text{ all } x > 0 \Rightarrow \{N_f(u_n)\}_{n \geq 1} \subseteq L^2(\Omega)\) is bounded. (29)

By [9], there exist \(\alpha \in (0, 1/2)\) and \(c_9 > 0\) such that \(u_n \in C^{0, \alpha}(\Omega)\), \(\|u_n\|_{C^{0, \alpha}(\Omega)} \leq c_9\) for all \(n \in \mathbb{N}\). Since \(C^{0, \alpha}(\Omega)\) is compactly embedded compactly in \(C(\Omega)\), we deduce that

\[u_n \rightarrow 0 \text{ in } C(\Omega).\] (30)

Recall that \(\|y_n\| = 1, \ y_n \geq 0\) for all \(n \in \mathbb{N}\). So, we may assume that

\[y_n \rightharpoonup y \text{ in } E_{\Sigma_1} \text{ and } y_n \rightarrow y \text{ in } L^2(\Omega), \ y \geq 0.\] (31)

It follows from (29), (30) and (31) that \(\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n \geq 1} \subseteq L^2(\Omega)\) is bounded. Thus, by hypothesis \(H(f)(iii)\), we have at least for a subsequence (see [11]),

\[\frac{N_f(u_n)}{\|u_n\|} \rightharpoonup \eta_0 y \text{ in } L^2(\Omega) \text{ with } 0 \leq \eta_0(z) \leq \hat{\eta} \text{ for almost all } z \in \Omega.\] (32)
We act on (28) with \( y_n - y \in E_{\Sigma_1} \) and pass to the limit as \( n \to \infty \). Using (30), (31) and (32) we obtain
\[
\lim_{n \to \infty} \langle A(y_n), y_n - y \rangle = 0.
\]
By the Kadec–Klee property we have \( y_n \to y \), hence \( \|y\| = 1 \), \( y \geq 0 \). In (28) we pass to the limit as \( n \to \infty \) and use (30), (32). Then
\[
A(y) = -\eta_0 y.
\]
Thus, by (32) we have
\[
\|Dy\|^2 = -\int_{\Omega} \eta_0 y^2 \, dz \leq 0,
\]
hence \( y = 0 \), a contradiction. This shows that the claim is true. Hence \( u_\ast \in S(\lambda_\ast) \subseteq E_{\Sigma_1} \cap C(\Omega) \) and so \( \lambda_\ast \in L \). □

Summarizing, we can state the following bifurcation-type theorem.

**Theorem 5.** If hypotheses \( H(f) \) hold, then there exists \( \lambda_\ast > 0 \) such that

(a) for all \( \lambda > \lambda_\ast \), problem \((P_\lambda)\) has at least two positive solutions \( u_0, \hat{u} \in E_{\Sigma_1} \cap C(\Omega) \);
(b) for \( \lambda = \lambda_\ast \), problem \((P_\lambda)\) has at least one positive solution \( u_\ast \in E_{\Sigma_1} \cap C(\Omega) \);
(c) for \( \lambda \in (0, \lambda_\ast) \), problem \((P_\lambda)\) has no positive solutions.

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