# ANISOTROPIC SINGULAR DOUBLE PHASE DIRICHLET PROBLEMS 

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#### Abstract

We consider an anisotropic double phase problem with a reaction in which we have the competing effects of a parametric singular term and a superlinear perturbation. We prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter varies on $\mathbb{R}_{+}=(0,+\infty)$. Our approach uses variational tools together with truncation and comparison techniques as well as several general results of independent interest about anisotropic equations, which are proved in the Appendix.


1. Features of the paper and historical comments. In this paper, we are concerned with the combined effects of an anisotropic differential operator, a parametric singular reaction, and a superlinear perturbation. The features of this paper are the following:
(i) the presence of a nonhomogeneous differential operator with different anisotropic growth, which generates a double phase associated energy;
(ii) the analysis developed in this paper is concerned with the combined effects of a nonstandard operator with unbalanced variable growth and a singular reaction;
(iii) the main result gives an exhaustive bifurcation picture according with a critical parameter;
(iv) our analysis combines the anisotropic nature of the differential operator, the singular nonlinearity with variable growth, and the superlinear perturbation term without satisfying the Ambrosetti-Rabinowitz condition.
[^0]To the best of our knowledge, this is the first paper dealing with anisotropic double phase problems with a singular reaction. We first recall some pioneering achievements in these fields.

We start with a short description on the development of double phase problems. To the best of our knowledge, the first contributions to this field are due to J. Ball [7], in relationship with problems in nonlinear elasticity and composite materials. Let $\Omega \subset \mathbb{R}^{N}$ be a bounded domain with smooth boundary. If $u: \Omega \mapsto \mathbb{R}^{N}$ is the displacement and if $D u$ is the $N \times N$ matrix of the deformation gradient, then the total energy can be represented by an integral of the type

$$
\begin{equation*}
I(u)=\int_{\Omega} f(z, D u(z)) d z \tag{1}
\end{equation*}
$$

where the energy function $f=f(z, \xi): \Omega \times \mathbb{R}^{N \times N} \mapsto \mathbb{R}$ is quasiconvex with respect to $\xi$. A simple example considered by Ball is given by functions $f$ of the type

$$
f(\xi)=g(\xi)+h(\operatorname{det} \xi)
$$

where $\operatorname{det} \xi$ is the determinant of the $N \times N$ matrix $\xi$, and $g, h$ are nonnegative convex functions, which satisfy the growth conditions

$$
g(\xi) \geq c_{1}|\xi|^{p} ; \quad \lim _{t \rightarrow+\infty} h(t)=+\infty
$$

where $c_{1}$ is a positive constant and $1<p<N$. The condition $p \leq N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the variational integral (1) that are discontinuous at one point where a cavity forms. In accordance with these problems arising in nonlinear elasticity, Marcellini [34, 35] considered continuous functions $f=f(z, u)$ with unbalanced growth that satisfy

$$
c_{1}|u|^{p} \leq|f(z, u)| \leq c_{2}\left(1+|u|^{q}\right) \quad \text { for all }(z, u) \in \Omega \times \mathbb{R}
$$

where $c_{1}, c_{2}$ are positive constants and $1 \leq p \leq q$.
A second feature of this paper consists in the presence of variable exponents, both in the expression of the differential operator that governs our problem and in the power-type nonlinear reaction. The study of differential equations and variational problems involving $p(z)$-growth conditions is a consequence of their applications. In 1920, Bingham was surprised that some paints do not run, like honey. He studied such a behaviour and described a strange phenomenon. There are fluids that flow then stop spontaneously (Bingham fluids). Within them, the forces that create flow reach a first threshold. As this threshold is not reached, the fluid flow without deforms as a solid. Invented in the 17 th century, the "Flemish medium" makes painting oil thixotropic: it fluids under pressure of the brush, but freezes as soon as you leave the rest. While the exact composition of the medium Flemish remains unknown, it is known that the bonds form gradually between its components, which is why the picture freezes in a few minutes. Thanks to this wonderful medium, Rubens have painted La Kermesse in 24 hours.

Materials requiring such more advanced theory have been studied experimentally since the middle of last century. The first major discovery on electrorheological fluids is due to Willis Winslow, who obtained a US patent on the effect in 1947 and wrote an article published in 1949, see [57]. These fluids have the interesting property that their viscosity depends on the electric field in the fluid. Winslow noticed that in such fluids (for instance lithium polymetachrylate) viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise
the viscosity by as much as five orders of magnitude. This phenomenon is known as the Winslow effect. For a general account of the underlying physics, we refer to Halsey [28].

Combining these features, the present paper considers a nonlinear problem whose associated functional contains the anisotropic double phase energy

$$
u \mapsto \int_{\Omega}\left(|D u|^{p(z)}+a(z)|D u(z)|^{q(z)}\right) d z
$$

where the modulating coefficient $a(z)$ dictates the geometry of the composite made by two differential materials, with hardening exponents $p(\cdot)$ and $q(\cdot)$, respectively. The isotropic case where $p$ and $q$ are constant functions was introduced by Zhikov [63] in the context of the Lavrentiev phenomenon.

Finally, we recall that singular problems have been intensively studied in the last decades. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, or in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction-diffusion processes where the solution $u \geq 0$ is viewed as the density of a reactant and the region where $u=0$ is called the dead core, where no reaction takes place. Nonlinear singular equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts, and in several other geophysical and industrial contents (see, e.g., the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence). Here we refer to the seminal paper by Crandall, Rabinowitz and Tartar [13] and to several subsequent works.
2. Introduction. Let $\Omega \subseteq \mathbb{R}^{N}$ be a bounded domain with $C^{2}$-boundary $\partial \Omega$. In this paper we study the following anisotropic singular double phase problem

$$
-\Delta_{p(z)} u(z)-\Delta_{q(z)} u(z)=\lambda u(z)^{-\eta(z)}+f(z, u(z)) \quad \text { in } \Omega,\left.u\right|_{\partial \Omega}=0, u>0
$$

In this problem, $p, q: \bar{\Omega} \mapsto(1,+\infty)$ are Lipschitz continuous functions satisfying $1<q_{-}=\min _{\bar{\Omega}} q \leq q_{+}=\max _{\bar{\Omega}} q<p_{-}=\min _{\bar{\Omega}} p \leq p_{+}=\max _{\bar{\Omega}} p$.

In general, given a function $r \in C(\bar{\Omega})$ with $1<r_{-}=\min _{\bar{\Omega}} r$, we denote by $\Delta_{r(z)}$ the anisotropic $r(z)$-Laplacian defined by

$$
\Delta_{r(z)} u=\operatorname{div}\left(|D u|^{r(z)-2} D u\right)
$$

for all $u \in W_{0}^{1, r(z)}(\Omega)$.
In the case of constant exponents (isotropic case), the double-phase problem $\left(P_{\lambda}\right)$ is motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [9] that appears in electromagnetism:

$$
-\operatorname{div}\left(\frac{D u}{\left(1-2|D u|^{2}\right)^{1 / 2}}\right)=h(u) \text { in } \Omega
$$

Indeed, by the Taylor formula, we have
$(1-x)^{-1 / 2}=1+\frac{x}{2}+\frac{3}{2 \cdot 2^{2}} x^{2}+\frac{5!!}{3!\cdot 2^{3}} x^{3}+\cdots+\frac{(2 n-3)!!}{(n-1)!2^{n-1}} x^{n-1}+\cdots$ for $|x|<1$.
Taking $x=2|D u|^{2}$ and adopting the first order approximation, we obtain problem $\left(P_{\lambda}\right)$ for $p=4$ and $q=2$. Furthermore, the $n$-th order approximation problem is
driven by the multi-phase differential operator

$$
-\Delta u-\Delta_{4} u-\frac{3}{2} \Delta_{6} u-\cdots-\frac{(2 n-3)!!}{(n-1)!} \Delta_{2 n} u
$$

In problem $\left(P_{\lambda}\right)$ the differential operator is the sum of two such variable exponent differential operators (anisotropic double phase problem). In the reaction (source) of problem $\left(P_{\lambda}\right)$ (that is, in the right-hand side of problem $\left(P_{\lambda}\right)$ ) we have the combined effects of a parametric singular term $\lambda u^{-\eta(z)}(\lambda>0$ being the parameter) and of a perturbation $f(z, x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$, the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, the function $x \mapsto$ $f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $\left(p_{+}-1\right)$-superlinear growth but without satisfying the Ambrosetti-Rabinowitz condition (the AR-condition for short), which is common in the literature when dealing with superlinear elliptic problems. In the singular term $\lambda u^{-\eta(z)}$, as we have already mentioned, $\lambda>0$ is a parameter and $\eta: \bar{\Omega} \mapsto \mathbb{R}$ is a Lipschitz continuous function satisfying $0<\eta(z)<1$ for all $z \in \bar{\Omega}$. We are looking for positive solutions of problem $\left(P_{\lambda}\right)$ and our goal is to describe the changes in the set of positive solutions as the parameter $\lambda$ moves on $\stackrel{\mathbb{R}}{+}^{+}=(0,+\infty)$. Finally, we prove the following bifurcation-type result. For the set of hypotheses $H_{0}, H_{1}$ mentioned in the theorem as well as the other notation used in the theorem, we refer to Section 3.

Theorem 2.1. If hypotheses $H_{0}, H_{1}$ hold, then there exists a critical parameter value $\lambda^{*}>0$ such that
(a) for every $\lambda \in\left(0, \lambda^{*}\right)$, problem $\left(P_{\lambda}\right)$ has at least two positive solutions $u_{0}$, $\widehat{u} \in \operatorname{int} C_{+}$;
(b) for $\lambda=\lambda^{*}$, problem $\left(P_{\lambda}\right)$ has at least one positive solution $u^{*} \in \operatorname{int} C_{+}$;
(c) for every $\lambda>\lambda^{*}$, problem $\left(P_{\lambda}\right)$ has no positive solution.

Our method of proof uses variational tools based on the critical point theory together with truncation and comparison techniques. For these techniques to work, we also prove some general results about anisotropic elliptic problems which are of independent interest. We have gathered all these results in the Appendix at the end of the paper. We believe that the results proved there will be useful to everyone working an anisotropic double phase problems.

The starting point of our work is the recent paper of Byun and Ko [11], which deals with anisotropic equations driven by the $p(z)$-Laplacian and with a reaction of the form $\lambda u^{-\eta(z)}+u^{q(z)-1}$, where $q \in C^{1}(\bar{\Omega})$ and $p_{+}<q(z)$ for all $z \in \bar{\Omega}$. The hypothesis on the exponent $p(\cdot)$ is stronger and it is required that $p \in C^{1}(\bar{\Omega})$ and it satisfies a kind of directional monotonicity condition (see hypothesis ( $p_{M}$ ) in [11]); this condition is motivated by the work of Fan, Zhang and Zhao [15]. To the best of our knowledge, the work of Byun and Ko [11] is the first one on anisotropic singular problems.

For isotropic equations, such problems were investigated by Ghergu and Rădulescu [23, 24], Haitao [27], Sun, Wu and Long [55] (semilinear equations driven by the Laplacian) and by Giacomoni, Schindler and Takač [25], Papageorgiou, Rădulescu and Repovš [40], Papageorgiou, Vetro and Vetro [45], Papageorgiou and Winkert [49]. Very recently, Papageorgiou, Rădulescu and Repovš [41] studied isotropic singular problems driven by a more general not necessary homogeneous, differential operator. Finally, we mention the work of Papageorgiou, Rădulescu and Repovš [43] on anisotropic equations driven by the $p(z)$-Laplacian plus an indefinite
potential and with a reaction in which we have the competition of "convex" and "concave" nonlinearities (anisotropic "concave-convex problem").

We mention that double phase problems arise in a variety of models of physical processes. We mention the works of Bahrouni, Rădulescu and Repovš [5, 6], Papageorgiou and Zhang [51], Ragusa and Tachikawa [53], Zhang and Rădulescu [60], Zhikov [63]. Singular problems arise in the study of cellular automata and interacting particle systems. Finally, we mention that in the last years isotropic double phase problems with unbalanced growth were studied systematically primarily by Mingione and co-workers, see [1, 8, 38]. We also mention the very recent work of Papageorgiou, Vetro and Vetro [47] on multiple solutions with sign information for parametric double phase problems with unbalanced growth. For related recent contributions to the study of anisotropic and singular problems in connection with models from the real world we refer to $[2,4,12,46,44,50,58,61,62]$.

One of the main difficulties we face in dealing with problem $\left(P_{\lambda}\right)$, is that the presence of the singular term $\lambda u^{-\eta(z)}$, leads to an energy (Euler) functional which is not $C^{1}$. So, we cannot apply to this variational integral the usual tools and results of critical point theory. We need to find ways to bypass the singular term and neutralize its effect.
3. Mathematical background and hypotheses. The study of problem $\left(P_{\lambda}\right)$ requires the use of spaces with variable exponent (Lebesgue and Sobolev spaces). A comprehensive treatment of such spaces can be found in the book of Diening, Harjulehto, Hästö and Růžička [14].

Let $\widehat{E}_{1}=\left\{r \in C(\bar{\Omega}): 1<\min _{\bar{\Omega}} r\right\}$. For any $r \in \widehat{E}_{1}$, we define

$$
r_{-}=\min _{\bar{\Omega}} r \text { and } r_{+}=\max _{\bar{\Omega}} r .
$$

Also, let $M(\Omega)=\{u: \Omega \mapsto \mathbb{R}$ measurable $\}$. As usual, we identify two such functions which differ only on a Lebesgue-null set.

Given $r \in \widehat{E}_{1}$, we define the variable exponent Lebesgue space as follows:

$$
L^{r(z)}(\Omega)=\left\{u \in M(\Omega): \int_{\Omega}|u|^{r(z)} d z<+\infty\right\} .
$$

We equip $L^{r(z)}(\Omega)$ with the so-called "Luxemburg norm" defined by

$$
\|u\|_{r(z)}=\inf \left\{t>0: \int_{\Omega}\left|\frac{u(z)}{t}\right|^{r(z)} d z \leq 1\right\} .
$$

Then $L^{r(z)}(\Omega)$ becomes a separable, reflexive (in fact, uniformly convex) Banach space. The dual of $L^{r(z)}(\Omega)$ is the space $L^{r^{\prime}(z)}(\Omega)$ with $r^{\prime} \in \widehat{E}_{1}$ satisfying $\frac{1}{r(z)}+$ $\frac{1}{r^{\prime}(z)}=1$ for all $z \in \bar{\Omega}$. We have the following Hölder-type inequality

$$
\left|\int_{\Omega} u h d z\right| \leq\left(\frac{1}{r_{-}}+\frac{1}{r_{-}^{\prime}}\right)\|u\|_{r(z)}\|h\|_{r^{\prime}(z)} .
$$

Moreover, if $r_{1}, r_{2} \in \widehat{E}_{1}$ satisfy $r_{1}(z) \leq r_{2}(z)$ for $z \in \bar{\Omega}$, then

$$
L^{r_{2}(z)}(\Omega) \hookrightarrow L^{r_{1}(z)}(\Omega) \text { continuously. }
$$

Using the variable exponent Lebesgue spaces, we can define the corresponding variable exponent Sobolev spaces. So, given $r \in \widehat{E}_{1}$ we define

$$
W^{1, r(z)}(\Omega)=\left\{u \in L^{r(z)}(\Omega):|D u| \in L^{r(z)}(\Omega)\right\}
$$

(as usual, the gradient $D u$ is understood in the weak sense).
We equip $W^{1, r(z)}(\Omega)$ with the following norm

$$
\|u\|_{1, r(z)}=\|u\|_{r(z)}+\|D u\|_{r(z)}
$$

for all $u \in W^{1, r(z)}(\Omega)$.
Suppose that $r \in \widehat{E}_{1}$ is Lipschitz continuous. Then we can define

$$
W_{0}^{1, r(z)}(\Omega)={\overline{C_{c}^{\infty}(\Omega)}}^{\|\cdot\|_{1, r(z)}}
$$

The spaces $W^{1, r(z)}(\Omega)$ and $W_{0}^{1, r(z)}(\Omega)$ are separable, reflexive (in fact, uniformly convex).

The critical Sobolev exponent is defined by

$$
r^{*}(z)= \begin{cases}\frac{N r(z)}{N-r(z)} & \text { if } r(z)<N \\ +\infty & \text { if } N \leq r(z)\end{cases}
$$

Suppose that $r, p \in C(\bar{\Omega}), 1<r_{-}, p_{+}<N$ and $1 \leq p(z) \leq r^{*}(z)$ for all $z \in \bar{\Omega}$ (resp., $1 \leq p(z)<r^{*}(z)$ for all $z \in \bar{\Omega}$ ). Then we have

$$
\begin{gathered}
W^{1, r(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega) \text { continuouly } \\
\left(\text { resp. } W^{1, r(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)\right. \text { compactly) }
\end{gathered}
$$

Similarly for $W_{0}^{1, q(z)}(\Omega)$, provided that $q(\cdot)$ is Lipschitz continuous, that is, $q \in$ $C^{0,1}(\bar{\Omega})$.

For $r \in C^{0,1}(\bar{\Omega})$, the Poincaré inequality holds, namely, there exists $c^{*}>0$ such that

$$
\|u\|_{r(z)} \leq c^{*}\|D u\|_{r(z)}
$$

for all $u \in W_{0}^{1, r(z)}(\Omega)$.
Therefore on $W_{0}^{1, p(z)}(\Omega)$ we can use the equivalent norm

$$
\|u\|_{1, r(z)}=\|D u\|_{r(z)}
$$

for all $u \in W_{0}^{1, r(z)}(\Omega)$.
The following modular function plays a central role in the study of these spaces. So, for $r \in \widehat{E}_{1}$, we define

$$
\rho_{r}(u)=\int_{\Omega}|u|^{r(z)} d z
$$

for all $u \in L^{r(z)}(\Omega)$.
Also, we define

$$
\rho_{r}(D u)=\rho_{r}(|D u|)
$$

for all $u \in W^{1, r(z)}(\Omega)$ or $W_{0}^{1, r(z)}(\Omega)$.
Sometimes when we want to emphasize the domain in $\mathbb{R}^{N}$ on which these modular functions are defined, we write

$$
\rho_{r}^{\Omega}(u) \quad \text { and } \quad \rho_{r}^{\Omega}(D u)
$$

Similarly for the norms. We write

$$
\|u\|_{r(z), \Omega} \quad \text { and } \quad\|u\|_{1, r(z), \Omega}
$$

The next proposition reveals the close relation between $\rho_{r}(\cdot)$ and the norm $\|\cdot\|_{r(z)}$.
Proposition 1. If $r \in \widehat{E}_{1}$ and $\left\{u, u_{n}\right\}_{n \geq 1} \subseteq L^{r(z)}(\Omega)$, then
(a) for all $t>0$ we have $\|u\|_{r(z)}=t$ if and only if $\rho_{r}\left(\frac{u}{t}\right)=1$;
(b) $\|u\|_{r(z)}<1($ resp. $=1,>1) \Leftrightarrow \rho_{r}(u)<1($ resp. $=1,>1)$;
(c) $\|u\|_{r(z)}<1 \Rightarrow\|u\|_{r(z)}^{r_{+}} \leq \rho_{r}(u) \leq\|u\|_{r(z)}^{r_{-}}$,
$\|u\|_{r(z)}>1 \Rightarrow\|u\|_{r(z)}^{r_{-}} \leq \rho_{r}(u) \leq\|u\|_{r(z)}^{r_{+}} ;$
(d) $\left\|u_{n}\right\|_{r(z)} \rightarrow 0 \Leftrightarrow \rho_{r}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$;
(e) $\left\|u_{n}\right\|_{r(z)} \rightarrow+\infty \Leftrightarrow \rho_{r}\left(u_{n}\right) \rightarrow+\infty$ as $n \rightarrow \infty$.

Let $r \in \widehat{E}_{1} \cap C^{0,1}(\bar{\Omega})$. Then we have

$$
W_{0}^{1, r(z)}(\Omega)^{*}=W^{-1, r^{\prime}(z)}(\Omega) .
$$

We define the operator $A_{r(z)}: W_{0}^{1, r(z)}(\Omega) \rightarrow W^{-1, r^{\prime}(z)}(\Omega)$ by

$$
\left\langle A_{r(z)}(u), h\right\rangle=\int_{\Omega}|D u|^{r(z)-2}(D u, D h)_{\mathbb{R}^{N}} d z
$$

for all $u, h \in W_{0}^{1, r(z)}(\Omega)$.
From Rădulescu and Repovš [52, p. 40] (see also Gasiński and Papageorgiou [20, Proposition 2.5]), we have the following properties.

Proposition 2. If $r \in \widehat{E}_{1} \cap C^{0,1}(\bar{\Omega})$, then the operator $A_{r(z)}: W_{0}^{1, r(z)}(\Omega) \mapsto$ $W^{-1, r^{\prime}(z)}(\Omega)$ is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too) and of type $(S)_{+}$, that is,

$$
\begin{gathered}
u_{n} \xrightarrow{w} u \text { in } W_{0}^{1, r(z)}(\Omega) \text { and } \limsup _{n \rightarrow \infty}\left\langle A_{r(z)}\left(u_{n}\right), u_{n}-u\right\rangle \leq 0 \\
\Downarrow \\
u_{n} \rightarrow u \text { in } W_{0}^{1, r(z)}(\Omega)
\end{gathered}
$$

Now let us introduce some basic notation which will be used in the rest of this paper.

For every $x \in \mathbb{R}$, we set $x^{ \pm}=\max \{ \pm x, 0\}$. Then given $u \in W_{0}^{1, r(z)}(\Omega)$ we define $u^{ \pm}(z)=u(z)^{ \pm}$for all $z \in \Omega$. We have

$$
u^{ \pm} \in W_{0}^{1, r(z)}(\Omega), u=u^{+}-u^{-},|u|=u^{+}+u^{-} .
$$

If $u, v \in W_{0}^{1, r(z)}(\Omega)$ and $u \leq v$, then we define

$$
\begin{aligned}
& {[u, v]=\left\{h \in W_{0}^{1, r(z)}(\Omega): u(z) \leq h(z) \leq v(z) \text { for a.a. } z \in \Omega\right\}} \\
& {[u)=\left\{h \in W_{0}^{1, r(z)}(\Omega): u(z) \leq h(z) \text { for a.a. } z \in \Omega\right\}}
\end{aligned}
$$

Let $C_{0}^{1}(\bar{\Omega})=\left\{u \in C^{1}(\bar{\Omega}):\left.u\right|_{\partial \Omega}=0\right\}$. We set

$$
\operatorname{int}_{C_{0}^{1}(\bar{\Omega})}[u, v]=\text { interior in } C_{0}^{1}(\bar{\Omega}) \text { of }[u, v]
$$

The space $C_{0}^{1}(\bar{\Omega})$ is an ordered Banach space, with positive (order) cone $C_{+}=$ $\left\{u \in C^{1}(\bar{\Omega}): u(z) \geq 0\right.$ for all $\left.z \in \bar{\Omega}\right\}$. This cone has a nonempty interior given by

$$
\operatorname{int} C_{+}=\left\{u \in C_{+}: u(z)>0 \text { for all } z \in \Omega,\left.\frac{\partial u}{\partial n}\right|_{\partial \Omega}<0\right\}
$$

with $n(\cdot)$ being the outward unit normal on $\partial \Omega$.
In $C^{1}(\bar{\Omega})$ we will also consider the following open cone

$$
D_{+}=\left\{u \in C^{1}(\bar{\Omega}): u(z)>0 \text { for all } z \in \Omega,\left.\frac{\partial u}{\partial n}\right|_{\partial \Omega \cap u^{-1}(0)}<0\right\}
$$

Suppose that $X$ is a Banach space and $\varphi \in C^{1}(X)$. We define

$$
K_{\varphi}=\left\{u \in X: \varphi^{\prime}(u)=0\right\}(\text { the critical set of } \varphi)
$$

We say that $\varphi(\cdot)$ satisfies the " $C$-condition", if the following property holds:
"Every sequence $\left\{u_{n}\right\}_{n \geq 1} \subseteq X$ which satisfies $\left\{\varphi\left(u_{n}\right)\right\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded,
and $\left(1+\left\|u_{n}\right\|_{X}\right) \varphi^{\prime}\left(u_{n}\right) \rightarrow 0$ in $X^{*}$ as $n \rightarrow \infty$, admits a strongly convergent subsequence".

This is a compactness-type of condition on $\varphi$. Since the space $X$ is not locally compact (being in general infinite dimensional), the burden of compactness is passed on $\varphi$. Using the $C$-condition, one can prove a deformation theorem from which follow the minimax theorems for the critical values of $\varphi$ (see Papageorgiou, Rădulescu and Repovš [42, Chapter 5]).

Now we will introduce our hypotheses on the data of problem $\left(P_{\lambda}\right)$.
$H_{0}: p, q, \eta \in C^{0,1}(\bar{\Omega})$ and $0<\eta(z)<1<q(z) \leq q_{+}<p_{-} \leq p(z)$ for all $z \in \bar{\Omega}$;
$H_{1}: f: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z, 0)=0$ for a.a. $z \in \Omega$ and
(i) $0 \leq f(z, x) \leq a(z)\left[1+x^{r(z)-1}\right]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $a \in L^{\infty}(\Omega)$, $r \in C(\bar{\Omega}), p(z)<r(z)<p_{-}^{*} \leq p^{*}(z)$ for all $z \in \bar{\Omega}$;
(ii) if $F(z, x)=\int_{0}^{x} f(z, s) d s$, then $\lim _{x \rightarrow+\infty} \frac{F(z, x)}{x^{p+}}=+\infty$ uniformly for a.a. $z \in \Omega ;$
(iii) there exists $\tau \in C(\bar{\Omega})$ such that

$$
\tau(z) \in\left(\left(r_{+}-p_{-}\right) \max \left\{\frac{N}{p_{-}}, 1\right\}, p_{+}^{*}\right) \text { for all } z \in \bar{\Omega}
$$

and $0<\beta_{0} \leq \liminf _{x \rightarrow+\infty} \frac{f(z, x) x-p_{+} F(z, x)}{x^{\tau(z)}}$ uniformly for a.a. $z \in \Omega$;
(iv) $\lim _{x \rightarrow 0^{+}} \frac{f(z, x)}{x^{q+-1}}=0$ uniformly for a.a. $z \in \Omega$;
(v) for every $\rho>0$, there exists $\widehat{\xi}_{\rho}>0$ such that for a.a. $z \in \Omega$, the function

$$
x \mapsto f(z, x)+\widehat{\xi}_{\rho} x^{p-1}
$$

is nondecreasing on $[0, \rho]$.

Remark 1. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_{+}=[0,+\infty)$, without any loss of generality, we assume that

$$
\begin{equation*}
f(z, x)=0 \text { for a.a. } z \in \Omega, \text { all } x \leq 0 \tag{2}
\end{equation*}
$$

Hypotheses $H_{1}($ ii $)$, (iii) imply that

$$
\lim _{x \rightarrow+\infty} \frac{f(z, x)}{x^{p_{+}-1}}=+\infty \text { uniformly for a.a. } z \in \Omega
$$

So, the perturbation $f(z, \cdot)$ is $\left(p_{+}-1\right)$-superlinear. Usually in the literature, superlinear problems are treated using the $A R$-condition (see Ambrosetti and Rabinowitz [3]). Here, we employ a less restrictive condition, namely hypothesis $H_{1}$ (iii), which incorporates in our framework superlinear perturbations with "slower" growth near $+\infty$. For example, the function

$$
f(z, x)=x^{p_{+}-1} \ln x+x^{p(z)-1} \text { for all } x>0
$$

with $p \in C^{0,1}(\bar{\Omega})$ such that $0 \leq \frac{1}{p_{-}}-\frac{1}{p_{+}}<\frac{1}{N}$, satisfies hypotheses $H_{1}$ but fails to satisfy the $A R$-condition.

For notational simplicity, in the sequel we denote by $\|\cdot\|$ the norm of the Sobolev space $W_{0}^{1, p(z)}(\Omega)$. Recall that by the Poincaré inequality, we have

$$
\|u\|=\|D u\|_{p(z)}
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
Also, for every $p \in \widehat{E}_{1}$, we write

$$
\rho_{p}(D u)=\rho_{p}(|D u|),
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
Given $r \in(1, \infty)$, we denote by $\widehat{u}_{1}(r)$ the positive, $L^{r}$-normalized (that is, $\left.\left\|\widehat{u}_{1}(r)\right\|_{r}=1\right)$ eigenfunction corresponding to the principal eigenvalue $\widehat{\lambda}_{1}(r)>0$ of $\left(-\Delta_{r}, W_{0}^{1, r}(\Omega)\right)$. Recall that $\widehat{\lambda}_{1}(r)>0$ is simple and isolated, and $\widehat{u}_{1}(r) \in \operatorname{int} C_{+}$ (see Gasiński and Papageorgiou [19, Section 6.2]).

Finally, we denote by $|\cdot|_{N}$ the Lebesgue measure on $\mathbb{R}^{N}$ and $\Omega_{0} \subset \subset \Omega$, means that $\bar{\Omega}_{0} \subset \Omega$.
4. Auxiliary results. As we already pointed out in the Introduction, we need to find ways to bypass the singular term and deal with $C^{1}$-functions. In this section we prove an auxiliary result which will be helpful in this direction.

So, we consider the following anisotropic purely singular Dirichlet problem

$$
-\Delta_{p(z)} u(z)-\Delta_{q(z)} u(z)=\lambda u(z)^{-\eta(z)} \quad \text { in } \Omega,\left.u\right|_{\partial \Omega}=0, u>0
$$

Proposition 3. If hypotheses $H_{0}$ hold and $\lambda>0$, then problem $\left(Q_{\lambda}\right)$ has a unique positive solution $\bar{u}_{\lambda} \in \operatorname{int} C_{+}$.

Proof. Let $h \in L^{p(z)}(\Omega)$ and $\varepsilon \in(0,1]$. We consider the following auxiliary Dirichlet problem

$$
\begin{equation*}
-\Delta_{p(z)} u(z)-\Delta_{q(z)} u(z)=\frac{\lambda}{[|h(z)|+\varepsilon]^{\eta(z)}} \quad \text { in } \Omega,\left.u\right|_{\partial \Omega}=0 \tag{3}
\end{equation*}
$$

The operator $V: W_{0}^{1, p(z)}(\Omega) \rightarrow W^{-1, p^{\prime}(z)}(\Omega)=W_{0}^{1, p(z)}(\Omega)^{*}$ defined by

$$
V(u)=A_{p(z)}(u)+A_{q(z)}(u) \text { for all } u \in W_{0}^{1, p(z)}(\Omega)
$$

is continuous, strictly monotone (hence maximal monotone, too) and coercive (see Propositions 2 and 1 ). Therefore $V(\cdot)$ is surjective (see Corollary 2.8.7 of Papageorgiou, Rădulescu and Repovš [42, p. 135]). Note that

$$
\begin{aligned}
& \left.0 \leq g_{\varepsilon}(z)=\frac{\lambda}{[|h(z)|+\varepsilon]^{\eta(z)}} \leq \frac{\lambda}{\varepsilon^{\eta_{+}}} \quad \text { (recall that } \varepsilon \in(0,1]\right) \\
\Rightarrow & g_{\varepsilon} \in L^{\infty}(\Omega)
\end{aligned}
$$

So, we can find $u_{\varepsilon} \in W_{0}^{1, p(z)}(\Omega), u_{\varepsilon} \neq 0$ such that

$$
\begin{equation*}
V\left(u_{\varepsilon}\right)=g_{\varepsilon} \tag{4}
\end{equation*}
$$

On account of the strict monotonicity of $V(\cdot)$, this solution is unique. Testing (4) with $-u_{\varepsilon}^{-} \in W_{0}^{1, p(z)}(\Omega)$, we obtain

$$
\begin{aligned}
& \frac{1}{p_{+}}\left[\rho_{p}\left(D u_{\varepsilon}^{-}\right)+\rho_{q}\left(D u_{\varepsilon}^{-}\right)\right] \leq 0 \\
\Rightarrow & u_{\varepsilon} \geq 0, u_{\varepsilon} \neq 0 \quad(\text { see Proposition 1) }
\end{aligned}
$$

From Theorem 4.1 of Fan and Zhao [16], we have that $u_{\varepsilon} \in L^{\infty}(\Omega)$. Then using Lemma 3.3 of Fukagai and Narukawa [17] (see also Lieberman [33] for isotropic problems), we obtain that $u_{\varepsilon} \in C_{+} \backslash\{0\}$. Invoking Proposition $A 2$ of the Appendix we infer that $u_{\varepsilon} \in \operatorname{int} C_{+}$.

We can define the solution map $\widehat{K}_{\varepsilon}: L^{p(z)}(\Omega) \mapsto L^{p(z)}(\Omega)$ by setting

$$
\widehat{K}_{\varepsilon}(h)=\widehat{u}_{\varepsilon} \text { for every } h \in L^{p(z)}(\Omega) .
$$

This map is well-defined. Let $\left\{h_{n}\right\}_{n \geq 1} \subseteq L^{p(z)}(\Omega)$ and assume that $h_{n} \rightarrow h$ in $L^{p(z)}(\Omega)$ as $n \rightarrow \infty$. Let $u_{n}=\widehat{K}_{\varepsilon}\left(h_{n}\right)$ for all $n \in \mathbb{N}$ and $u=\widehat{K}_{\varepsilon}(h)$. We have

$$
\begin{equation*}
\left\langle A_{p(z)}\left(u_{n}\right), g\right\rangle+\left\langle A_{q(z)}\left(u_{n}\right), g\right\rangle=\int_{\Omega} \frac{1}{\left[\left|h_{n}\right|+\varepsilon\right]^{\eta(z)}} g d z \tag{5}
\end{equation*}
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$.
Choosing $g=u_{n} \in W_{0}^{1, p(z)}(\Omega)$ in (5), we see that

$$
\left\{u_{n}\right\}_{n \geq 1} \subseteq W_{0}^{1, p(z)}(\Omega) \text { is bounded. }
$$

So, we may assume that

$$
\begin{equation*}
u_{n} \xrightarrow{w} \widetilde{u} \text { in } W_{0}^{1, p(z)}(\Omega) \text { and } u_{n} \rightarrow \widetilde{u} \text { in } L^{p(z)}(\Omega) . \tag{6}
\end{equation*}
$$

In (5) we choose $g=u_{n}-\widetilde{u} \in W_{0}^{1, p(z)}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (6). We obtain

$$
\begin{aligned}
& \lim _{n \rightarrow \infty}\left[\left\langle A_{p(z)}\left(u_{n}\right), u_{n}-\widetilde{u}\right\rangle+\left\langle A_{q(z)}\left(u_{n}\right), u_{n}-\widetilde{u}\right\rangle\right]=0, \\
\Rightarrow & \limsup _{n \rightarrow \infty}\left[\left\langle A_{p(z)}\left(u_{n}\right), u_{n}-\widetilde{u}\right\rangle+\left\langle A_{q(z)}(\widetilde{u}), u_{n}-\widetilde{u}\right\rangle\right] \leq 0
\end{aligned}
$$

(since $A_{q(z)}(\cdot)$ is monotone),
$\Rightarrow \limsup _{n \rightarrow \infty}\left\langle A_{p(z)}\left(u_{n}\right), u_{n}-\widetilde{u}\right\rangle \leq 0$,
$\Rightarrow u_{n} \rightarrow \widetilde{u}$ in $W_{0}^{1, p(z)}(\Omega) \quad$ (see Proposition 2).

If in (5) we pass to the limit as $n \rightarrow \infty$ and use (7), then

$$
\begin{aligned}
& \left\langle A_{p(z)}(\widetilde{u}), g\right\rangle+\left\langle A_{q(z)}(\widetilde{u}), g\right\rangle=\int_{\Omega} \frac{1}{[|h|+\varepsilon]^{\eta(z)}} d z \\
& \text { for all } g \in W_{0}^{1, p(z)}(\Omega), \\
\Rightarrow & \widetilde{u}=u, \\
\Rightarrow & \widehat{K}_{\varepsilon}(\cdot) \text { is continuous. }
\end{aligned}
$$

From the above argument we see that $\widehat{K}_{\varepsilon}\left(L^{p(z)}(\Omega)\right)$ is a bounded set in $W_{0}^{1, p(z)}(\Omega)$. But $W_{0}^{1, p(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)$ compactly. So, by the Schauder fixed point theorem (see Theorem 3.2.20 of Papageorgiou, Rădulescu and Repovš [42, p. 197]) we can find $u_{\varepsilon}^{*} \in W_{0}^{1, p(z)}(\Omega)$ such that $u_{\varepsilon}^{*}=\widehat{K}_{\varepsilon}\left(u_{\varepsilon}^{*}\right)$. As above, the anisotropic regularity theory and Proposition $A 2$, imply that $u_{\varepsilon}^{*} \in \operatorname{int} C_{+}$. Moreover, from the monotonicity of $A_{p(z)}(\cdot), A_{q(z)}(\cdot)$ and the fact that the mapping $x \mapsto \frac{1}{(x+\varepsilon)^{\eta(z)}}$ is decreasing on $\mathbb{R}_{+}$we infer that this solution $u_{\varepsilon}^{*} \in \operatorname{int} C_{+}$is unique.
Claim. $0<\varepsilon^{\prime}<\varepsilon \Rightarrow u_{\varepsilon}^{*} \leq u_{\varepsilon^{\prime}}^{*}$.
Note that

$$
\begin{align*}
-\Delta_{p(z)} u_{\varepsilon^{\prime}}^{*}(z)-\Delta_{q(z)} u_{\varepsilon^{\prime}}^{*}(z) & =\frac{\lambda}{\left[u_{\varepsilon^{\prime}}^{*}(z)+\varepsilon^{\prime}\right]} \\
& \left.\geq \frac{\lambda}{\left[u_{\varepsilon}^{*}(z)+\varepsilon\right]} \quad \text { in } \Omega \text { (recall that } \varepsilon^{\prime}<\varepsilon\right) . \tag{8}
\end{align*}
$$

We consider the Carathéodory function $l_{\varepsilon}: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ defined by

$$
l_{\varepsilon}(z, x)= \begin{cases}\frac{\lambda}{\left(x^{+}+\varepsilon\right)^{\eta(z)}} & \text { if } x \leq u_{\varepsilon^{\prime}}^{*}(z),  \tag{9}\\ \frac{\lambda}{\left[u_{\varepsilon^{\prime}}^{*}(z)+\varepsilon\right]^{\eta(z)}} & \text { if } u_{\varepsilon^{\prime}}^{*}(z)<x .\end{cases}
$$

We set $L_{\varepsilon}(z, x)=\int_{0}^{x} l_{\varepsilon}(z, s) d s$ and consider the $C^{1}$-functional $\gamma_{\varepsilon}: W_{0}^{1, p(z)}(\Omega) \mapsto$ $\mathbb{R}$ defined by

$$
\gamma_{\varepsilon}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}|D u|^{q(z)} d z-\int_{\Omega} l_{\varepsilon}(z, u) d z,
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
From (9) it is clear that $\gamma_{\varepsilon}(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widetilde{u}_{\varepsilon} \in$ $W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{equation*}
\gamma_{\varepsilon}\left(\widetilde{u}_{\varepsilon}\right)=\min \left\{\gamma_{\varepsilon}(u): u \in W_{0}^{1, p(z)}(\Omega)\right\} . \tag{10}
\end{equation*}
$$

Let $u \in \operatorname{int} C_{+}$. Since $u_{\varepsilon^{\prime}}^{*} \in \operatorname{int} C_{+}$, we can find $t \in(0,1)$ small such that $t u \leq \min \left\{u_{\varepsilon^{\prime}}^{*}, \varepsilon\right\}$ (see Proposition 4.1.22 of Papageorgiou, Rădulescu and Repovš [42, p. 274]). We have

$$
\gamma_{\varepsilon}(t u) \leq \frac{t^{q_{-}}}{q_{-}}\left[\rho_{p}(D u)+\rho_{q}(D u)\right]-\frac{\lambda t}{(2 \varepsilon)^{\eta_{+}}}|\Omega|_{N} \quad(\text { see }(9)) .
$$

Since $1<q_{-}$, choosing $t \in(0,1)$ even smaller if necessary, we have

$$
\begin{aligned}
& \gamma_{\varepsilon}(t u)<0 \\
\Rightarrow & \gamma_{\varepsilon}\left(\widetilde{u}_{\varepsilon}\right)<0=\gamma_{\varepsilon}(0) \quad(\text { see }(10)), \\
\Rightarrow & \widetilde{u}_{\varepsilon} \neq 0
\end{aligned}
$$

From (10) we have

$$
\begin{align*}
& \gamma_{\varepsilon}^{\prime}\left(\widetilde{u}_{\varepsilon}\right)=0 \\
\Rightarrow & \left\langle A_{p(z)}\left(\widetilde{u}_{\varepsilon}\right), g\right\rangle+\left\langle A_{q(z)}\left(\widetilde{u}_{\varepsilon}\right), g\right\rangle=\int_{\Omega} l_{\varepsilon}\left(z, \widetilde{u}_{\varepsilon}\right) g d z  \tag{11}\\
& \text { for all } g \in W_{0}^{1, p(z)}(\Omega)
\end{align*}
$$

In relation (11), we first choose $g=-\widetilde{u}_{\varepsilon}^{-} \in W_{0}^{1, p(z)}(\Omega)$. We obtain

$$
\widetilde{u}_{\varepsilon} \geq 0, \widetilde{u}_{\varepsilon} \neq 0
$$

Also, if in (11) we choose $g=\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon^{\prime}}^{*}\right)^{+} \in W_{0}^{1, p(z)}(\Omega)$, then we have

$$
\begin{aligned}
& \left\langle A_{p(z)}\left(\widetilde{u}_{\varepsilon}\right),\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon^{\prime}}^{*}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(\widetilde{u}_{\varepsilon}\right),\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon^{\prime}}^{*}\right)^{+}\right\rangle \\
= & \int_{\Omega} \frac{\lambda}{\left(u_{\varepsilon^{\prime}}^{*}+\varepsilon\right)^{\eta(z)}}\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon^{\prime}}^{*}\right)^{+} d z \quad(\text { see }(9)) \\
\leq & \left\langle A_{p(z)}\left(u_{\varepsilon^{\prime}}^{*}\right),\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon^{\prime}}^{*}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(u_{\varepsilon^{\prime}}^{*}\right),\left(\widetilde{u}_{\varepsilon}-u_{\varepsilon^{\prime}}^{*}\right)^{+}\right\rangle \quad(\text { see }(8)), \\
\Rightarrow & \widetilde{u}_{\varepsilon} \leq u_{\varepsilon^{\prime}}^{*}
\end{aligned}
$$

So, we have proved that

$$
\begin{aligned}
& \widetilde{u}_{\varepsilon} \in\left[0, u_{\varepsilon^{\prime}}^{*}\right], \widetilde{u}_{\varepsilon} \neq 0, \\
\Rightarrow & \widetilde{u}_{\varepsilon}=u_{\varepsilon}^{*} \quad(\operatorname{see}(9)), \\
\Rightarrow & u_{\varepsilon}^{*} \leq u_{\varepsilon^{\prime}}^{*}
\end{aligned}
$$

This proves the Claim.
Now let $\varepsilon_{n}=\frac{1}{n}$ and $u_{n}^{*}=u_{\varepsilon_{n}}^{*} \in \operatorname{int} C_{+}$for all $n \in \mathbb{N}$. We have

$$
-\Delta_{p(z)} u_{n}^{*}(z)-\Delta_{q(z)} u_{n}^{*}(z)=\frac{\lambda}{\left[u_{n}^{*}(z)+\frac{1}{n}\right]^{\eta(z)}} \quad \text { in } \Omega \text { for all } n \in \mathbb{N} \text {. }
$$

Testing this equation with $u_{n}^{*} \in W_{0}^{1, p(z)}(\Omega)$, we obtain

$$
\begin{aligned}
& \rho_{p}\left(D u_{n}^{*}\right)+\rho_{q}\left(D u_{n}^{*}\right)=\int_{\Omega} \frac{\lambda u_{n}^{*}}{\left(u_{n}^{*}+\frac{1}{n}\right)^{\eta(z)}} d z \leq \lambda \int_{\Omega} u_{n}^{* 1-\eta(z)} d z \quad \text { for all } n \in \mathbb{N}, \\
\Rightarrow & \left\{u_{n}^{*}\right\}_{n \geq 1} \subseteq W_{0}^{1, p(z)}(\Omega) \text { is bounded (see Proposition 1). }
\end{aligned}
$$

So, we may assume that

$$
u_{n}^{*} \xrightarrow{w} \bar{u}_{\lambda} \text { in } W_{0}^{1, p(z)}(\Omega) \text { and } u_{n}^{*} \rightarrow \bar{u}_{\lambda} \text { in } L^{p(z)}(\Omega) \text { as } n \rightarrow \infty .
$$

For every $n \in \mathbb{N}$, we have

$$
\begin{aligned}
& \left\langle A_{p(z)}\left(u_{n}^{*}\right), u_{n}^{*}-\bar{u}_{\lambda}\right\rangle+\left\langle A_{q(z)}\left(u_{n}^{*}\right), u_{n}^{*}-\bar{u}_{\lambda}\right\rangle \\
= & \int_{\Omega} \frac{\lambda\left(u_{n}^{*}-\bar{u}_{\lambda}\right)}{\left(u_{n}^{*}+\frac{1}{n}\right)^{\eta(z)}} d z \\
\leq & \int_{\Omega} \frac{\lambda\left(u_{n}^{*}-\bar{u}_{\lambda}\right)}{u_{1}^{* \eta(z)}} d z \quad \text { (see the Claim). }
\end{aligned}
$$

On account of Lemma 14.16 of Gilbarg and Trudinger [26, p. 335], we see that $\widehat{d}(\cdot)=d(\cdot, \partial \Omega) \in C_{+}$. So, we can find $c_{1}>0$ such that $\widehat{d} \leq c_{1} u_{1}^{*}$. Then we have

$$
\begin{aligned}
\frac{\lambda\left(u_{n}^{*}-\bar{u}_{\lambda}\right)}{u_{1}^{* \eta(z)}} & \leq \lambda\left(u_{1}^{*}\right)^{1-\eta(z)} \frac{u_{n}^{*}}{u_{1}^{*}} \\
& \leq \lambda c_{2} \frac{u_{n}^{*}}{\widehat{d}} \text { for some } c_{2}>0, \text { all } n \in \mathbb{N}
\end{aligned}
$$

On account of the anisotropic Hardy inequality of Harjulehto, Hästö and Koskenoja [29] and of [15], we infer that $\left\{\frac{u_{n}^{*}}{\vec{d}}\right\}_{n \in \mathbb{N}}$ is uniformly integrable.

Therefore by Vitali's theorem, we have

$$
\begin{align*}
& \int_{\Omega} \frac{\lambda\left(u_{n}^{*}-\bar{u}_{\lambda}\right)}{\left(u_{n}^{*}\right)^{\eta(z)}} d z \rightarrow 0 \text { as } n \rightarrow \infty \\
\Rightarrow & \limsup _{n \rightarrow \infty}\left[\left\langle A_{p(z)}\left(u_{n}^{*}\right), u_{n}^{*}-\bar{u}_{\lambda}\right\rangle+\left\langle A_{q(z)}\left(u_{n}^{*}\right), u_{n}^{*}-\bar{u}_{\lambda}\right\rangle\right] \leq 0 \\
\Rightarrow & \left.u_{n}^{*} \rightarrow \bar{u}_{\lambda} \text { in } W_{0}^{1, p(z)}(\Omega) \text { as } n \rightarrow \infty \quad \text { (as before, see }(7)\right) . \tag{12}
\end{align*}
$$

For every $n \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\langle A_{p(z)}\left(u_{n}^{*}\right), g\right\rangle+\left\langle A_{q(z)}\left(u_{n}^{*}\right), g\right\rangle=\int_{\Omega} \frac{\lambda g}{\left(u_{n}^{*}+\frac{1}{n}\right)^{\eta(z)}} d z \tag{13}
\end{equation*}
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$.
Passing to the limit as $n \rightarrow \infty$ in (13) and using (12), we obtain

$$
\begin{equation*}
\left\langle A_{p(z)}\left(\bar{u}_{\lambda}\right), g\right\rangle+\left\langle A_{q(z)}\left(\bar{u}_{\lambda}\right), g\right\rangle=\int_{\Omega} \frac{\lambda g}{\bar{u}_{\lambda}{ }^{\eta(z)}} d z \tag{14}
\end{equation*}
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$.
It is easy to check that $\left(u_{1}^{*}\right)^{-\eta(z)} \in L^{s}(\Omega)$ for $s>N$. We know that $u_{1}^{*} \leq u_{n}^{*}$ for all $n \in \mathbb{N}$ (see the Claim), hence $u_{1}^{*} \leq \bar{u}_{\lambda}$, which implies that $\bar{u}_{\lambda}{ }^{-\eta(z)} \in L^{s}(\Omega)$, $s>N$. Then from (14) we obtain that $\bar{u}_{\lambda}$ is a positive solution of problem $\left(Q_{\lambda}\right)$. By Proposition $A 1$ of the Appendix we have that $\bar{u}_{\lambda} \in L^{\infty}(\Omega)$ and then from Lemma 3.3 of Fukagai and Narukawa [17] we deduce that $\bar{u}_{\lambda} \in C_{+} \backslash\{0\}$. Moreover, Proposition $A 2$ of the Appendix implies that $\bar{u}_{\lambda} \in \operatorname{int} C_{+}$.

Finally, we show that this solution is unique. So, suppose that $\bar{v}_{\lambda}$ is another positive solution of problem $\left(Q_{\lambda}\right)$. Again we have $\bar{v}_{\lambda} \in \operatorname{int} C_{+}$. Exploiting the strictly monotonicity of $A_{p(z)}(\cdot)$ and of $A_{q(z)}(\cdot)$, we have

$$
\begin{aligned}
0 & \leq\left\langle A_{p(z)}\left(\bar{u}_{\lambda}\right)-A_{p(z)}\left(\bar{v}_{\lambda}\right), \bar{u}_{\lambda}-\bar{v}_{\lambda}\right\rangle+\left\langle A_{q(z)}\left(\bar{u}_{\lambda}\right)-A_{q(z)}\left(\bar{v}_{\lambda}\right), \bar{u}_{\lambda}-\bar{v}_{\lambda}\right\rangle \\
& =\int_{\Omega} \lambda\left[\frac{1}{\bar{u}_{\lambda}^{\eta(z)}}-\frac{1}{\bar{v}_{\lambda}{ }^{\eta(z)}}\right]\left(\bar{u}_{\lambda}-\bar{v}_{\lambda}\right) d z \leq 0, \\
\Rightarrow \bar{u}_{\lambda} & =\bar{v}_{\lambda} .
\end{aligned}
$$

The proof is now complete.
We introduce the following two sets

$$
\begin{aligned}
\mathfrak{L}= & \left\{\lambda>0: \text { problem }\left(P_{\lambda}\right) \text { has a positive solution }\right\} \\
& (\text { that is, } \mathfrak{L} \text { is the set of admissible parameters) }, \\
S_{\lambda}= & \text { set of positive solutions of problem }\left(P_{\lambda}\right) .
\end{aligned}
$$

In the next section we establish the structural and regularity properties of these two sets.
5. Properties of the sets $\mathfrak{L}$ and $S_{\lambda}$. We start by showing the nonemptiness of set $\mathfrak{L}$, that is, there exist admissible parameters.

Proposition 4. If hypotheses $H_{0}, H_{1}$ hold, then $\mathfrak{L} \neq \emptyset$.
Proof. Let $\lambda>0$ and let $\bar{u}_{\lambda} \in \operatorname{int} C_{+}$be the unique positive solution of problem $\left(Q_{\lambda}\right)$ produced in Proposition 3. We consider the Carathéodory function $k_{\lambda}(z, x)$ defined by

$$
\begin{align*}
& k_{\lambda}(z, x)= \begin{cases}\lambda \bar{u}_{\lambda}(z)^{-\eta(z)}+f\left(z, x^{+}\right) & \text {if } x \leq \bar{u}_{\lambda}(z) \\
\lambda x^{-\eta(z)}+f(z, x) & \text { if } \bar{u}_{\lambda}(z)<x\end{cases}  \tag{15}\\
& \text { (notice } \bar{u}_{\lambda}^{-\eta(\cdot)} \in L^{1}(\Omega), \text { see Lazer and McKenna [32]). }
\end{align*}
$$

We set $K_{\lambda}(z, x)=\int_{0}^{x} k_{\lambda}(z, s) d s$ and introduce the $C^{1}$-functional $\psi_{\lambda}: W_{0}^{1, p(z)}(\Omega)$ $\mapsto \mathbb{R}$ defined by

$$
\psi_{\lambda}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}|D u|^{q(z)} d z-\int_{\Omega} K_{\lambda}(z, u) d z
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
On account of hypotheses $H_{1}(\mathrm{i})$, (iv), we see that given $\varepsilon>0$, we can find $c_{\varepsilon}>0$ such that

$$
\begin{equation*}
F(z, x) \leq \frac{\varepsilon}{q_{+}} x^{q_{+}}+c_{\varepsilon} x^{r_{+}} \text {for a.a. } z \in \Omega, \text { all } x \geq 0 \tag{16}
\end{equation*}
$$

Let $u \in W_{0}^{1, p(z)}(\Omega)$ with $\|u\| \leq 1$. We have

$$
\begin{aligned}
\psi_{\lambda}(u) \geq & \frac{1}{q_{+}} \rho_{q}(D u)-\int_{\left\{0 \leq u \leq \bar{u}_{\lambda}\right\}}\left[\lambda \bar{u}_{\lambda}{ }^{-\eta(z)} u+F\left(z, u^{+}\right)\right] d z \\
- & \frac{\lambda}{1-\eta_{+}} \int_{\left\{u>\bar{u}_{\lambda}\right\}}\left[u^{1-\eta(z)}-\bar{u}_{\lambda}^{1-\eta(z)}\right] d z \\
- & \int_{\left\{u>\bar{u}_{\lambda}\right\}}\left[F(z, u)-F\left(z, \bar{u}_{\lambda}\right)\right] d z \quad(\text { see }(15)) \\
\geq & \frac{1}{q_{+}}\|u\|^{q_{+}}-\frac{\lambda}{1-\eta_{+}} \int_{\Omega}\left(u^{+}\right)^{1-\eta(z)} d z-\int_{\Omega} F\left(z, u^{+}\right) d z \\
& \quad(\text { since }\|u\| \leq 1, F \geq 0 \text { and see }(2)) \\
\geq & \frac{1}{q_{+}}\left(1-\varepsilon c_{5}\right)\|u\|^{q_{+}}-c_{6}\|u\|^{r_{+}-\lambda c_{7}\|u\|^{1-\eta_{+}}} \begin{aligned}
& \text { for some } c_{5}>0, c_{6}=c_{6}(\varepsilon)>0 \text { and } c_{7}>0(\text { see }(16))
\end{aligned}
\end{aligned}
$$

Choosing $\varepsilon \in\left(0, \frac{1}{c_{5}}\right)$, we obtain

$$
\begin{equation*}
\psi_{\lambda}(u) \geq c_{8}\|u\|^{q_{+}}-c_{6}\|u\|^{r_{+}}-\lambda c_{7}\|u\|^{1-\eta_{+}} \tag{17}
\end{equation*}
$$

for some $c_{8}>0$.
Recall that $q_{+}<r_{+}$. So, we can find $\rho \in(0,1)$ small such that

$$
\begin{equation*}
c_{8} \rho^{q_{+}}-c_{6} \rho^{r_{+}} \geq \widehat{\eta}_{0}>0 \tag{18}
\end{equation*}
$$

Then let $\lambda_{0}>0$ be small so that

$$
\begin{equation*}
\lambda c_{7} \rho^{1-\eta_{+}} \leq \frac{1}{2} \widehat{\eta}_{0} \text { for all } \lambda \in\left(0, \lambda_{0}\right] \tag{19}
\end{equation*}
$$

Using (18) and (19) in (17) we obtain

$$
\begin{equation*}
\psi_{\lambda}(u) \geq \frac{1}{2} \widehat{\eta}_{0}>0 \tag{20}
\end{equation*}
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$ with $\|u\|=\rho$, all $0<\lambda \leq \lambda_{0}$.
Moreover, on account of hypothesis $H_{1}(i i)$, if $\widetilde{u} \in \operatorname{int} C_{+}$, then

$$
\begin{equation*}
\psi_{\lambda}(t \widetilde{u}) \rightarrow-\infty \text { as } t \rightarrow+\infty \tag{21}
\end{equation*}
$$

Claim. For every $\lambda>0$, the functional $\psi_{\lambda}(\cdot)$ satisfies the $C$-condition.
Let $\left\{u_{n}\right\}_{n \geq 1} \subseteq W_{0}^{1, p(z)}(\Omega)$ be a sequence such that

$$
\begin{align*}
& \left|\psi_{\lambda}\left(u_{n}\right)\right| \leq c_{9} \text { for some } c_{9}>0, \text { all } n \in \mathbb{N}  \tag{22}\\
& \left(1+\left\|u_{n}\right\|\right) \psi_{\lambda}^{\prime}\left(u_{n}\right) \rightarrow 0 \text { in } W^{-1, p^{\prime}(z)}(\Omega) \text { as } n \rightarrow \infty \tag{23}
\end{align*}
$$

From (23) we have

$$
\begin{equation*}
\left|\left\langle A_{p(z)}\left(u_{n}\right), g\right\rangle+\left\langle A_{q(z)}\left(u_{n}\right), g\right\rangle-\int_{\Omega} k_{\lambda}\left(z, u_{n}\right) g d z\right| \leq \frac{\varepsilon_{n}\|g\|}{1+\left\|u_{n}\right\|}, \tag{24}
\end{equation*}
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$, with $\varepsilon_{n} \rightarrow 0^{+}$.
In (24) we choose $g=-u_{n}^{-} \in W_{0}^{1, p(z)}(\Omega)$. We obtain

$$
\begin{align*}
& \rho_{p}\left(D u_{n}^{-}\right)+\rho_{q}\left(D u_{n}^{-}\right) \leq c_{10}\left\|u_{n}^{-}\right\| \\
& \text {for some } c_{10}>0, \text { all } n \in \mathbb{N}(\text { see (15) and (2)) } \\
\Rightarrow & \left\{u_{n}^{-}\right\}_{n \geq 1} \subseteq W_{0}^{1, p(z)}(\Omega) \text { is bounded (see Proposition 1). } \tag{25}
\end{align*}
$$

Next, we choose $g=u_{n}^{+} \in W_{0}^{1, p(z)}(\Omega)$ in (24). We obtain

$$
\begin{equation*}
-\rho_{p}\left(D u_{n}^{+}\right)-\rho_{q}\left(D u_{n}^{+}\right)+\int_{\Omega} k_{\lambda}\left(z, u_{n}^{+}\right) u_{n}^{+} d z \leq \varepsilon_{n} \tag{26}
\end{equation*}
$$

for all $n \in \mathbb{N}$.
From (22) and (25), we have

$$
\begin{align*}
& \int_{\Omega} \frac{1}{p(z)}\left|D_{n}^{+}\right|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}\left|D_{n}^{+}\right|^{q(z)} d z-\int_{\Omega} K_{\lambda}\left(z, u_{n}^{+}\right) d z \leq c_{11} \\
\Rightarrow & \frac{1}{p_{+}}\left[\rho_{p}\left(D u_{n}^{+}\right)+\rho_{q}\left(D u_{n}^{+}\right)-\int_{\Omega} p_{+} K_{\lambda}\left(z, u_{n}^{+}\right) d z\right] \leq c_{11} \\
\Rightarrow & \rho_{p}\left(D u_{n}^{+}\right)+\rho_{q}\left(D u_{n}^{+}\right)-\int_{\Omega} p_{+} K_{\lambda}\left(z, u_{n}^{+}\right) d z \leq p_{+} c_{11} \tag{27}
\end{align*}
$$

for some $c_{11}>0$, all $n \in \mathbb{N}$.

We add (26) and (27) and obtain

$$
\begin{align*}
& \int_{\Omega}\left[k_{\lambda}\left(z, u_{n}^{+}\right) u_{n}^{+}-p_{+} K_{\lambda}\left(z, u_{n}^{+}\right)\right] d z \leq c_{12} \\
\Rightarrow & \int_{\Omega}\left[f\left(z, u_{n}^{+}\right) u_{n}^{+}-p_{+} F\left(z, u_{n}^{+}\right)\right] d z \leq c_{13}\left[1+\int_{\Omega}\left(u_{n}^{+}\right)^{1-\eta(z)} d z\right] \tag{28}
\end{align*}
$$

for some $c_{12}, c_{13}>0$, all $n \in \mathbb{N}$.
On account of hypotheses $H_{1}(\mathrm{i})$, (iii), we can find $\beta_{1} \in\left(0, \beta_{0}\right)$ and $c_{14}>0$ such that

$$
\begin{equation*}
\beta_{1} x^{\tau(z)}-c_{14} \leq f(z, x) x-p_{+} F(z, x) \text { for a.a. } z \in \Omega, \text { all } x \geq 0 \tag{29}
\end{equation*}
$$

We use (29) in (28) and obtain

$$
\begin{align*}
& \rho_{\tau}\left(u_{n}^{+}\right) \leq c_{15}\left[1+\left\|u_{n}^{+}\right\|_{\tau(z)}\right] \text { for some } c_{15}>0, \text { all } n \in \mathbb{N}, \\
\Rightarrow & \left\{u_{n}^{+}\right\}_{n \geq 1} \subseteq L^{\tau(z)}(\Omega) \hookrightarrow L^{\tau_{-}}(\Omega) \text { is bounded. } \tag{30}
\end{align*}
$$

It is clear from hypothesis $H_{1}$ (iii) that without any loss of generality, we may assume that

$$
\begin{aligned}
& \tau(z)<r(z)<p^{*}(z) \\
\Rightarrow & \tau_{-}<r_{+}<p_{-}^{*} \quad\left(\text { see hypothesis } H_{1}(\mathrm{i})\right)
\end{aligned}
$$

Hence we can find $t \in(0,1)$ such that

$$
\begin{equation*}
\frac{1}{r_{+}}=\frac{1-t}{\tau_{-}}+\frac{t}{p_{-}^{*}} \tag{31}
\end{equation*}
$$

Invoking the interpolation inequality (see Proposition 2.3.17 of Papageorgiou and Winkert [48, p. 116]), we obtain

$$
\begin{align*}
& \left\|u_{n}^{+}\right\|_{r_{+}} \leq\left\|u_{n}^{+}\right\|_{\tau_{-}}^{1-t}\left\|u_{-}^{+}\right\|_{p_{-}^{*}}^{t} \text { for all } n \in \mathbb{N}, \\
\Rightarrow & \left\|u_{n}^{+}\right\|_{r_{+}}^{r_{+}} \leq c_{16}\left\|u_{n}\right\|^{t r_{+}} \text {for some } c_{16}>0, \text { all } n \in \mathbb{N} \quad(\text { see }(30)) \tag{32}
\end{align*}
$$

From (24) with $h=u_{n}^{+} \in W_{0}^{1, p(z)}(\Omega)$, we have

$$
\begin{align*}
\rho_{p}\left(D u_{n}^{+}\right)+\rho_{q}\left(D u_{n}^{+}\right) & \leq \varepsilon_{n}+\int_{\Omega} k_{\lambda}\left(z, u_{n}^{+}\right) u_{n}^{+} d z \\
\Rightarrow \rho_{p}\left(D u_{n}^{+}\right)+\rho_{q}\left(D u_{n}^{+}\right) & \leq c_{17}\left[1+\int_{\Omega} f\left(z, u_{n}^{+}\right) u_{n}^{+} d z\right] \\
& \leq c_{18}\left(1+\left\|u_{n}^{+}\right\|_{r_{+}}^{r_{+}}\right) \quad\left(\text { see hypothesis } H_{1}(\mathrm{i})\right) \\
& \leq c_{19}\left(1+\left\|u_{n}\right\|^{t_{+}}\right) \quad(\text { see }(32)), \tag{33}
\end{align*}
$$

for some $c_{17}, c_{18}, c_{19}>0$, all $n \in \mathbb{N}$.
First suppose that $p_{-}^{*} \neq N$. We have

$$
p_{-}^{*}=\frac{N p_{-}}{N-p_{-}} \text {if } p_{-}<N \text { and } p_{-}^{*}=+\infty \text { if } p_{-}>N
$$

From (31) we have

$$
t r_{+}=\frac{p_{-}^{*}\left(r_{+}-\tau_{-}\right)}{p_{-}^{*}-\tau_{-}} \text {if } p_{-}<N \text { and } t r_{+}=r_{+}-\tau_{-} \text {if } N<p_{-}
$$

Using hypothesis $H_{1}$ (iii), we see that

$$
\begin{equation*}
t r_{+}<p_{-} \tag{34}
\end{equation*}
$$

From (34), (33) and Proposition 1 it follows that

$$
\begin{equation*}
\left\{u_{n}^{+}\right\}_{n \geq 1} \subseteq W_{0}^{1, p(z)}(\Omega) \text { is bounded. } \tag{35}
\end{equation*}
$$

Next, suppose that $p_{-}=N$. Then $p_{-}^{*}=+\infty$ and we know that $W_{0}^{1, p(z)}(\Omega) \hookrightarrow$ $L^{\theta}(\Omega)$ for all $1 \leq \theta<\infty$. Then for the above argument to work, we need to consider $\theta>r_{+}>\tau_{-}$. As before, let $t \in(0,1)$ such that

$$
\begin{aligned}
\frac{1}{r_{+}} & =\frac{1-t}{\tau_{-}}+\frac{t}{\theta} \\
\Rightarrow t r_{+} & =\frac{\theta\left(r_{+}-\tau_{-}\right)}{\theta-\tau_{-}} \rightarrow r_{+}-\tau_{-}<p_{-} \text {as } \theta \rightarrow+\infty
\end{aligned}
$$

(see hypothesis $H_{1}(\mathrm{iii})$ ).
So, we choose $\theta>r_{+}$big such that

$$
t r_{+}<p_{-}
$$

Then from this and (33) again we obtain (35).
From (25) and (35), we see that $\left\{u_{n}\right\}_{n \geq 1} \subset W_{0}^{1, p(z)}(\Omega)$ is bounded and so we may assume that

$$
\begin{equation*}
u_{n} \xrightarrow{w} u \text { in } W_{0}^{1, p(z)}(\Omega) \text { and } u_{n} \rightarrow u \text { in } L^{r(z)}(\Omega) . \tag{36}
\end{equation*}
$$

In (24) we use $g=u_{n}-u \in W_{0}^{1, p(z)}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (36). Then as in the proof of Proposition 3, we obtain

$$
\begin{aligned}
& u_{n} \rightarrow u \text { in } W_{0}^{1, p(z)}(\Omega) \quad(\text { see }(7)) \\
\Rightarrow & \psi_{\lambda}(\cdot) \text { satisfies the } C \text {-condition. }
\end{aligned}
$$

This proves the Claim.
From (20), (21) and the Claim, we see that we can apply the mountain pass theorem. So, for every $\lambda \in\left(0, \lambda_{0}\right]$, we can find $u_{\lambda} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{equation*}
u_{\lambda} \in K_{\psi_{\lambda}} \text { and } \psi_{\lambda}(0)=0<\frac{1}{q} \widehat{\eta}_{0} \leq \psi_{\lambda}\left(u_{\lambda}\right) \quad(\text { see }(20)) \tag{37}
\end{equation*}
$$

From (37) we have $u_{\lambda} \neq 0$ and

$$
\begin{equation*}
\left\langle A_{p(z)}\left(u_{\lambda}\right), g\right\rangle+\left\langle A_{q(z)}\left(u_{\lambda}\right), g\right\rangle=\int_{\Omega} k_{\lambda}\left(z, u_{\lambda}\right) g d z \tag{38}
\end{equation*}
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$.
In (38) we choose $g=\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+} \in W_{0}^{1, p(z)}(\Omega)$ and have

$$
\begin{aligned}
& \left\langle A_{p(z)}\left(u_{\lambda}\right),\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(u_{\lambda}\right),\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+}\right\rangle \\
= & \int_{\Omega}\left[\lambda \bar{u}_{\lambda}{ }^{-\eta(z)}+f\left(z, u_{\lambda}^{+}\right)\right]\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+} d z \quad(\text { see }(15)) \\
\geq & \int_{\Omega} \lambda \bar{u}_{\lambda}{ }^{-\eta(z)}\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+} d z \quad\left(\text { see hypothesis } H_{1}(\mathrm{i})\right) \\
= & \left\langle A_{p(z)}\left(\bar{u}_{\lambda}\right),\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(\bar{u}_{\lambda}\right),\left(\bar{u}_{\lambda}-u_{\lambda}\right)^{+}\right\rangle \quad(\text { see Proposition 3) }, \\
\Rightarrow & \bar{u}_{\lambda} \leq u_{\lambda}, \\
\Rightarrow & u_{\lambda} \in S_{\lambda} \quad(\text { see }(24) \text { and }(15)) \text { and so }\left(0, \lambda_{0}\right) \subseteq \mathfrak{L} \neq \emptyset .
\end{aligned}
$$

The proof is now complete.

Next, we determine the regularity properties of the elements of $S_{\lambda}$. This will also lead to important structural properties of $\mathfrak{L}$.

We start by obtaining a lower bound for the elements of $S_{\lambda}$.
Proposition 5. If hypotheses $H_{0}, H_{1}$ hold, then $\bar{u}_{\lambda} \leq u$ for all $u \in S_{\lambda}$.
Proof. Let $u \in S_{\lambda}$ and consider the following function defined on $\Omega \times \stackrel{\circ}{\mathbb{R}}_{+}=\Omega \times$ $(0,+\infty)$

$$
e_{\lambda}(z, x)= \begin{cases}\lambda x^{-\eta(z)} & \text { if } 0<x \leq u(z)  \tag{39}\\ \lambda u(z)^{-\eta(z)} & \text { if } u(z)<x\end{cases}
$$

This is a Carathéodory function on $\Omega \times \mathbb{R}_{+}$. We consider the following anisotropic, singular double phase problem

$$
-\Delta_{p(z)} u(z)-\Delta_{q(z)} u(z)-e_{\lambda}(z, u(z)) \text { in } \Omega,\left.u\right|_{\partial \Omega}=0, u>0
$$

As in the proof of Proposition 3, via a fixed point argument, we show that for every $\lambda>0$, problem $\left(T_{\lambda}\right)$ admits a positive solution $\widetilde{u}_{\lambda} \in \operatorname{int} C_{+}$.

We have

$$
\begin{aligned}
& \left\langle A_{p(z)}\left(\widetilde{u}_{\lambda}\right),\left(\widetilde{u}_{\lambda}-u\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(\widetilde{u}_{\lambda}\right),\left(\widetilde{u}_{\lambda}-u\right)^{+}\right\rangle \\
= & \lambda \int_{\Omega} u^{-\eta(z)}\left(\widetilde{u}_{\lambda}-u\right)^{+} d z \quad(\text { see }(39)) \\
\leq & \int_{\Omega}\left[\lambda u^{-\eta(z)}+f(z, u)\right]\left(\widetilde{u}_{\lambda}-u\right)^{+} d z \quad\left(\text { see hypothesis } H_{1}(\mathrm{i})\right) \\
= & \left\langle A_{p(z)}(u),\left(\widetilde{u}_{\lambda}-u\right)^{+}\right\rangle+\left\langle A_{q(z)}(u),\left(\widetilde{u}_{\lambda}-u\right)^{+}\right\rangle \quad\left(\text { since } u \in S_{\lambda}\right), \\
\Rightarrow & \widetilde{u}_{\lambda} \leq u, \\
\Rightarrow & \widetilde{u}_{\lambda}=\bar{u}_{\lambda} \quad(\text { see }(39) \text { and Proposition } 3), \\
\Rightarrow & \bar{u}_{\lambda} \leq u \quad \text { for all } u \in S_{\lambda} .
\end{aligned}
$$

The proof is now complete.
If in the above proof we replace $\lambda>0$ by $\mu \in(0, \lambda)$ and $u$ by $\bar{u}_{\lambda} \in \operatorname{int} C_{+}$, then we derive the following monotonicity property of the solution map $\lambda \mapsto \bar{u}_{\lambda}(\lambda>0)$ of problem $\left(Q_{\lambda}\right)$ (see Proposition 3).
Proposition 6. If hypotheses $H_{0}$ hold and $0<\mu<\lambda$, then $\bar{u}_{\mu} \leq \bar{u}_{\lambda}$.
As a consequence of Proposition 5 and Theorem B. 1 of Saoudi and Ghanmi [54] we have the following regularity result for $S_{\lambda}$.

Proposition 7. If hypotheses $H_{0}, H_{1}$ hold and $\lambda \in \mathfrak{L}$, then $S_{\lambda} \subseteq\left[\bar{u}_{\lambda}\right) \cap \operatorname{int} C_{+}$.
Proof. Let $u \in S_{\lambda}$. From Proposition 5, we have

$$
\begin{align*}
0 & \leq \bar{u}_{\lambda} \leq u \\
\Rightarrow 0 & \leq u^{-\eta(z)} \leq \bar{u}_{\lambda}{ }^{-\eta(z)} \in L^{s}(\Omega), s>N \tag{40}
\end{align*}
$$

(see the proof of Proposition 3).
Also, from Proposition $A 1$ of the Appendix, we have

$$
\begin{equation*}
u \in L^{\infty}(\Omega) \tag{41}
\end{equation*}
$$

We know that

$$
\begin{equation*}
-\Delta_{p(z)} u(z)-\Delta_{q(z)} u(z)=\lambda u(z)^{-\eta(z)}+f(z, u(z)) \quad \text { in } \Omega \tag{42}
\end{equation*}
$$

Let $g(\cdot)=\lambda u(\cdot)^{-\eta(z)} \in L^{s}(\Omega), s>N($ see (40)) and consider the following linear Dirichlet problem

$$
\begin{equation*}
-\Delta v(z)=g(z) \text { in } \Omega,\left.v\right|_{\partial \Omega}=0 \tag{43}
\end{equation*}
$$

Theorem 9.15 of Gilbarg and Trudinger [26, p. 241] implies that problem (43) has a unique solution $v \in W^{2, s}(\Omega)$. The Sobolev embedding theorem implies that $W^{2, s}(\Omega) \hookrightarrow C^{1, \alpha}(\bar{\Omega})$ with $\alpha=1-\frac{N}{s} \in(0,1)$ (recall that $s>N$ ). Let $\xi=D v \in$ $C^{0, \alpha}\left(\bar{\Omega}, \mathbb{R}^{N}\right)$. By (41) and hypothesis $H_{1}(\mathrm{i})$ we obtain that

$$
\theta(\cdot)=f(\cdot, u(\cdot)) \in L^{\infty}(\Omega)
$$

We rewrite (42) as follows

$$
-\operatorname{div}\left(|D u|^{p(z)-2} D u+|D u|^{q(z)-2} D u-\xi\right)=\theta \quad \text { in } \Omega,\left.u\right|_{\partial \Omega}=0
$$

Then Lemma 3.3 of Fukagai and Narukawa [17] implies that

$$
\begin{aligned}
& u \in C_{0}^{1, \alpha}(\bar{\Omega})=C^{1, \alpha}(\bar{\Omega}) \cap C_{0}^{1}(\bar{\Omega}) \\
\Rightarrow & u \in\left[\bar{u}_{\lambda}\right) \cap \operatorname{int} C_{+} \quad(\text { see Proposition } 5) .
\end{aligned}
$$

The proof is now complete.
Now we can prove a structural property for the set $\mathfrak{L}$. Namely, we show that $\mathfrak{L}$ is an interval.

Proposition 8. If hypotheses $H_{0}, H_{1}$ hold, $\lambda \in \mathfrak{L}$ and $\mu \in(0, \lambda)$, then $\mu \in \mathfrak{L}$.
Proof. Since $\lambda \in \mathfrak{L}$, we can find $u_{\lambda} \in S_{\lambda} \subseteq \operatorname{int} C_{+}$(see Proposition 7). By Proposition 6 we have $\bar{u}_{\mu} \leq \bar{u}_{\lambda}$. Therefore $\bar{u}_{\mu} \leq u_{\lambda}$ and so we can define the Carathéodory function $j_{\mu}(z, x)$ by

$$
j_{\mu}(z, x)= \begin{cases}\mu \bar{u}_{\mu}(z)^{-\eta(z)}+f\left(z, \bar{u}_{\mu}(z)\right) & \text { if } x<\bar{u}_{\mu}(z)  \tag{44}\\ \mu x^{-\eta(z)}+f(z, x) & \text { if } \bar{u}_{\mu}(z) \leq x \leq u_{\lambda}(z) \\ \mu u_{\lambda}(z)^{-\eta(z)}+f\left(z, u_{\lambda}(z)\right) & \text { if } u_{\lambda}(z)<x\end{cases}
$$

We set $J_{\mu}(z, x)=\int_{0}^{x} j_{\mu}(z, s) d s$ and introduce the $C^{1}$-functional $\sigma_{\mu}: W_{0}^{1, p(z)}(\Omega)$ $\mapsto \mathbb{R}$ defined by

$$
\sigma_{\mu}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}|D u|^{q(z)} d z-\int_{\Omega} J_{\mu}(z, u) d z
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
By (44) and Proposition 1, it is clear that $\sigma_{\mu}(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem we can find $u_{\mu} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{align*}
& \sigma_{\mu}\left(u_{\mu}\right)=\min \left\{\sigma_{\mu}(u): u \in W_{0}^{1, p(z)}(\Omega)\right\} \\
\Rightarrow & \sigma_{\mu}^{\prime}\left(u_{\mu}\right)=0, \\
\Rightarrow & \left\langle A_{p(z)}\left(u_{\mu}\right), g\right\rangle+\left\langle A_{q(z)}\left(u_{\mu}\right), g\right\rangle=\int_{\Omega} j_{\mu}\left(z, u_{\mu}\right) g d z  \tag{45}\\
& \text { for all } g \in W_{0}^{1, p(z)}(\Omega) .
\end{align*}
$$

We test (45) with $h=\left(\bar{u}_{\mu}-u_{\mu}\right)^{+} \in W_{0}^{1, p(z)}(\Omega)$. We have

$$
\begin{aligned}
& \left\langle A_{p(z)}\left(u_{\mu}\right),\left(\bar{u}_{\mu}-u_{\mu}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(u_{\mu}\right),\left(\bar{u}_{\mu}-u_{\mu}\right)^{+}\right\rangle \\
= & \int_{\Omega}\left[\mu \bar{u}_{\mu}^{-\eta(z)}+f\left(z, \bar{u}_{\lambda}\right)\right]\left(\bar{u}_{\mu}-u_{\mu}\right)^{+} d z \quad(\text { see }(44)) \\
\geq & \int_{\Omega} \mu \bar{u}_{\mu}^{-\eta(z)} d z \quad\left(\text { see hypothesis } H_{1}(\mathrm{i})\right) \\
= & \left\langle A_{p(z)}\left(\bar{u}_{\mu}\right),\left(\bar{u}_{\mu}-u_{\mu}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(\bar{u}_{\mu}\right),\left(\bar{u}_{\mu}-u_{\mu}\right)^{+}\right\rangle \\
\Rightarrow & \bar{u}_{\mu} \leq u_{\mu}
\end{aligned}
$$

Next, we choose $h=\left(u_{\mu}-u_{\lambda}\right)^{+} \in W_{0}^{1, p(z)}(\Omega)$ in (45). Then

$$
\begin{aligned}
& \left\langle A_{p(z)}\left(u_{\mu}\right),\left(u_{\mu}-u_{\lambda}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(u_{\mu}\right),\left(u_{\mu}-u_{\lambda}\right)^{+}\right\rangle \\
= & \int_{\Omega}\left[\mu u_{\lambda}^{-\eta(z)}+f\left(z, u_{\lambda}\right)\right]\left(u_{\mu}-u_{\lambda}\right)^{+} d z \quad(\text { see }(44)) \\
\leq & \int_{\Omega}\left[\lambda u_{\lambda}-\eta(z)+f\left(z, u_{\lambda}\right)\right]\left(u_{\mu}-u_{\lambda}\right)^{+} d z \quad(\text { since } \mu<\lambda) \\
= & \left\langle A_{p(z)}\left(u_{\lambda}\right),\left(u_{\mu}-u_{\lambda}\right)^{+}\right\rangle+\left\langle A_{q(z)}\left(u_{\lambda}\right),\left(u_{\mu}-u_{\lambda}\right)^{+}\right\rangle \quad\left(\text { since } u_{\lambda} \in S_{\lambda}\right), \\
\Rightarrow & u_{\mu} \leq u_{\lambda}
\end{aligned}
$$

So, we have proved that

$$
\begin{equation*}
u_{\mu} \in\left[\bar{u}_{\mu}, u_{\lambda}\right] \tag{46}
\end{equation*}
$$

From (46), (44) and (45) it follows that

$$
u_{\mu} \in S_{\mu} \subseteq\left[\bar{u}_{\mu}\right) \cap \operatorname{int} C_{+}(\text {see Proposition } 7), \text { hence } \mu \in \mathfrak{L} .
$$

The proof is now complete.
A byproduct of the above proof, is the following weak monotonicity property of the solution multifunction $\lambda \mapsto S_{\lambda}$.

Proposition 9. If hypotheses $H_{0}, H_{1}$ hold, $\lambda \in \mathfrak{L}$, $u_{\lambda} \in S_{\lambda}$ and $\mu \in(0, \lambda)$, then $\mu \in \mathfrak{L}$ and we can find $u_{\mu} \in S_{\mu}$ such that $u_{\mu} \leq u_{\lambda}$.

In fact, using Proposition $A 4$ of the Appendix (the anisotropic strong comparison principle), we can have a stronger version of Proposition 9.

Proposition 10. If hypotheses $H_{0}, H_{1}$ hold, $\lambda \in \mathfrak{L}, u_{\lambda} \in S_{\lambda}$ and $\mu \in(0, \lambda)$, then $\mu \in \mathfrak{L}$ and there exists $u_{\mu} \in S_{\mu}$ such that

$$
u_{\lambda}-u_{\mu} \in \operatorname{int} C_{+}
$$

Proof. From Proposition 9, we already know that $\mu \in \mathfrak{L}$ and we can find $u_{\mu} \in S_{\mu} \subseteq$ int $C_{+}$such that

$$
\begin{equation*}
0 \leq u_{\mu} \leq u_{\lambda} \tag{47}
\end{equation*}
$$

Let $\rho=\left\|u_{\lambda}\right\|_{\infty}$ (recall that $u_{\lambda} \in S_{\lambda} \subseteq \operatorname{int} C_{+}$) and let $\widehat{\xi}_{\rho}>0$ be as postulated by hypothesis $H_{1}(\mathrm{v})$. We have

$$
\begin{aligned}
& -\Delta_{p(z)} u_{\mu}-\Delta_{q(z)} u_{\mu}+\widehat{\xi}_{\rho} u_{\rho}{ }^{p(z)-1}-\lambda u_{\mu}{ }^{-\eta(z)} \\
= & f\left(z, u_{\mu}\right)+\widehat{\xi}_{\rho} u_{\mu}^{p(z)-1}-(\lambda-\mu) u_{\mu}^{-\eta(z)} \quad\left(\text { since } u_{\mu} \in S_{\mu}\right)
\end{aligned}
$$

$$
\begin{aligned}
\leq & f\left(z, u_{\mu}\right)+\widehat{\xi}_{\rho} u_{\mu}^{p(z)-1}-(\lambda-\mu)\left\|u_{\mu}\right\|_{\infty}^{-\eta(z)} \\
& \left(\text { see }(47), \text { hypothesis } H_{1}(\mathrm{v}) \text { and recall that } \mu<\lambda\right) \\
\leq & -\Delta_{p(z)} u_{\lambda}-\Delta_{q(z)} u_{\lambda}+\widehat{\xi}_{\rho} u_{\lambda}{ }^{-\eta(z)} \quad \text { in } \Omega
\end{aligned}
$$

Then Proposition $A 4$ of the Appendix implies that

$$
u_{\lambda}-u_{\mu} \in \operatorname{int} C_{+}
$$

The proof is now complete.
Let $\lambda^{*}=\sup \mathfrak{L}$.
Proposition 11. If hypotheses $H_{0}, H_{1}$ hold, then $\lambda^{*} \in \mathfrak{L}$.
Proof. Hypotheses $H_{1}$ imply that we can find $\widehat{\lambda}>0$ such that

$$
\begin{equation*}
\frac{\widehat{\lambda}}{2} x^{-\eta(z)}+f(z, x) \geq x^{p(z)-1} \text { for a.a. } z \in \Omega, \text { all } x \geq 0 \tag{48}
\end{equation*}
$$

Let $\lambda>\widehat{\lambda}$ and suppose that $\lambda \in \mathfrak{L}$. Then we can find $u_{\lambda} \in S_{\lambda} \subseteq \operatorname{int} C_{+}$. Let $\Omega_{0} \subset \subset \Omega$ with $C^{2}$-boundary $\partial \Omega_{0}$. We set

$$
0<m_{\lambda}^{0}=\min _{\bar{\Omega}_{0}} u_{\lambda} \leq 1
$$

(recall that $u_{\lambda} \in \operatorname{int} C_{+}$and choose $\Omega_{0}$ such that $d\left(\bar{\Omega}_{0}, \partial \Omega\right)$ is small).
For $\delta \in\left(0, m_{\lambda}^{0}\right)$, we set $\left(m_{\lambda}^{0}\right)^{\delta}=m_{\lambda}^{0}+\delta$. For $\rho=\left\|u_{\lambda}\right\|_{\infty}$, let $\widehat{\xi}_{\rho}>0$ be as postulated by hypothesis $H_{1}(\mathrm{v})$. We have

$$
\begin{align*}
\lambda\left[\left(m_{\lambda}^{0}\right)^{\delta}\right]^{-\eta(z)} & =\frac{\lambda}{\left(m_{\lambda}^{0}+\delta\right)^{\eta(z)}} \\
& \geq \frac{\lambda}{\left(m_{\lambda}^{0}\right)^{\eta(z)}+\delta^{\eta(z)}} \quad(\text { since } 0<\eta(z)<1 \text { for all } z \in \bar{\Omega}) \\
& \left.\geq \frac{\lambda}{2} \frac{1}{\left(m_{\lambda}^{0}\right)^{\eta(z)}} \quad \text { (recall that } 0<\delta<m_{\lambda}^{0}\right) \tag{49}
\end{align*}
$$

We have

$$
\begin{aligned}
& -\Delta_{p(z)}\left(m_{\lambda}^{0}\right)^{\delta}-\Delta_{q(z)}\left(m_{\lambda}^{0}\right)^{\delta}+\widehat{\xi}_{\rho}\left[\left(m_{\lambda}^{0}\right)^{\delta}\right]^{p(z)-1}-\lambda\left[\left(m_{\lambda}^{0}\right)^{\delta}\right]^{-\eta(z)} \\
\leq & \widehat{\xi}_{\rho}\left(m_{\lambda}^{0}\right)^{p(z)-1}+\chi(\delta)-\frac{\lambda}{2} \frac{1}{\left(m_{\lambda}^{0}\right)^{\eta(z)}} \quad \text { with } \chi(\delta) \rightarrow 0^{+} \text {as } \delta \rightarrow 0^{+} \quad(\text { see }(49)) \\
\leq & \left(\widehat{\xi}_{\rho}+1\right)\left(m_{\lambda}^{0}\right)^{p(z)-1}+\chi(\delta)-\frac{\lambda}{2} \frac{1}{\left(m_{\lambda}^{0}\right)^{\eta(z)}} \\
\leq & f\left(z, m_{\lambda}^{0}\right)+\widehat{\xi}_{\rho}\left(m_{\lambda}^{0}\right)^{p(z)-1}-\frac{1}{2}(\lambda-\widehat{\lambda}) \frac{1}{\left(m_{\lambda}^{0}\right)^{\eta(z)}}+\chi(\delta) \quad(\text { see }(48)) \\
\leq & f\left(z, m_{\lambda}^{0}\right)+\widehat{\xi}_{\rho}\left(m_{\lambda}^{0}\right)^{p(z)-1}-\frac{1}{2}(\lambda-\widehat{\lambda}) \frac{1}{\left(m_{\lambda}^{0}\right)^{\eta-}}+\chi(\delta) \quad\left(\text { since } m_{\lambda}^{0} \leq 1, \widehat{\lambda}<\lambda\right) \\
< & f\left(z, m_{\lambda}^{0}\right)+\widehat{\xi}_{\rho}\left(m_{\lambda}^{0}\right)^{p(z)-1} \quad \text { for } \delta \in\left(0, m_{\lambda}^{0}\right) \text { small } \\
= & -\Delta_{p(z)} u_{\lambda}-\Delta_{q(z)} u_{\lambda}+\widehat{\xi}_{\rho} u_{\lambda}^{p(z)-1}-\lambda u_{\lambda}^{-\eta(z)} \quad \text { in } \Omega_{0} .
\end{aligned}
$$

Invoking Proposition $A 4$ of the Appendix, we have

$$
\left(m_{\lambda}^{0}\right)^{\delta}<u(z) \text { for all } z \in \Omega_{0}, \text { all } \lambda \in\left(0, m_{\lambda}^{0}\right) \text { small }
$$

a contradiction to the definition of $m_{\lambda}^{0}$. Therefore $\lambda \notin \mathfrak{L}$ and so $\lambda^{*} \leq \widehat{\lambda}<\infty$.

The proof is now complete.
So far we have that

$$
\left(0, \lambda^{*}\right) \subseteq \mathfrak{L} \subseteq\left(0, \lambda^{*}\right]
$$

6. Multiple positive solutions and the critical parameter $\lambda^{*}$. In this section we show that for $\lambda \in\left(0, \lambda^{*}\right)$ we have multiplicity of the positive solutions and finally we show that the critical parameter value is admissible. In this way we complete the proof of Theorem 2.1.

Proposition 12. If hypotheses $H_{0}, H_{1}$ hold and $\lambda \in\left(0, \lambda^{*}\right)$, then problem $\left(P_{\lambda}\right)$ has at least two positive solutions

$$
u_{0}, \widehat{u} \in \operatorname{int} C_{+}, u_{0} \neq \widehat{u}
$$

Proof. Let $0<\mu<\lambda<\theta<\lambda^{*}$. We know that $\mu, \theta \in \mathfrak{L}$ (see Proposition 8). Also, on account of Proposition 10, we can find $u_{\theta} \in S_{\theta} \subseteq \operatorname{int} C_{+}, u_{0} \in S_{\lambda} \subseteq \operatorname{int} C_{+}$and $u_{\mu} \in S_{\mu} \subseteq \operatorname{int} C_{+}$such that

$$
\begin{align*}
& u_{0}-u_{\mu} \in \operatorname{int} C_{+} \text {and } u_{\theta}-u_{0} \in \operatorname{int} C_{+}, \\
\Rightarrow & u_{0} \in \operatorname{int}_{C_{0}^{1}(\bar{\Omega})}\left[u_{\mu}, u_{\theta}\right] . \tag{50}
\end{align*}
$$

We consider the Carathéodory function $i_{\lambda}(z, x)$ defined by

$$
i_{\lambda}(z, x)= \begin{cases}\lambda u_{\mu}(z)^{-\eta(z)}+f\left(z, u_{\mu}(z)\right) & \text { if } x \leq u_{\mu}(z)  \tag{51}\\ \lambda x^{-\eta(z)}+f(z, x) & \text { if } u_{\mu}(z)<x\end{cases}
$$

Also, we consider the following truncation of $i_{\lambda}(z, \cdot)$

$$
\widehat{i}_{\lambda}(z, x)= \begin{cases}i_{\lambda}(z, x) & \text { if } x \leq u_{\theta}(z)  \tag{52}\\ i_{\lambda}\left(z, u_{\theta}(z)\right) & \text { if } u_{\theta}(z)<x\end{cases}
$$

This is also a Carathéodory function. We set

$$
I_{\lambda}(z, x)=\int_{0}^{x} i_{\lambda}(z, s) d s \text { and } \widehat{I}_{\lambda}(z, x)=\int_{0}^{x} \widehat{i}_{\lambda}(z, s) d s
$$

and consider the $C^{1}$-functionals $d_{\lambda}, \widehat{d}_{\lambda}: W_{0}^{1, p(z)}(\Omega) \mapsto \mathbb{R}$ defined by

$$
\begin{aligned}
& d_{\lambda}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}|D u|^{q(z)} d z-\int_{\Omega} I_{\lambda}(z, u) d z \\
& \widehat{d}_{\lambda}(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}|D u|^{q(z)} d z-\int_{\Omega} \widehat{I}_{\lambda}(z, u) d z
\end{aligned}
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
Using (51) and (52), we can show that

$$
\begin{equation*}
K_{d_{\lambda}} \subseteq\left[u_{\mu}\right) \cap \operatorname{int} C_{+} \text {and } K_{\widehat{d}_{\lambda}} \subseteq\left[u_{\mu}, u_{\theta}\right] \cap \operatorname{int} C_{+} \tag{53}
\end{equation*}
$$

From (53) we see that without any loss of generality, we may assume that

$$
\begin{equation*}
K_{d_{\lambda}} \text { is finite and } K_{\widehat{d}_{\lambda}}=\left\{u_{0}\right\} \tag{54}
\end{equation*}
$$

Otherwise, on account of (53), (51) and (52), problem ( $P_{\lambda}$ ) already has two positive solutions and so we are done.

Note that

$$
\begin{equation*}
\left.d_{\lambda}\right|_{\left[u_{\mu}, u_{\theta}\right]}=\left.\widehat{d}_{\lambda}\right|_{\left[u_{\mu}, u_{\theta}\right]} \text { and }\left.d_{\lambda}^{\prime}\right|_{\left[u_{\mu}, u_{\theta}\right]}=\left.\widehat{d}_{\lambda}^{\prime}\right|_{\left[u_{\mu}, u_{\theta}\right]} . \tag{55}
\end{equation*}
$$

It is clear that $\widehat{d}_{\lambda}(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_{0} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{aligned}
& \widehat{d}_{\lambda}\left(\widetilde{u}_{0}\right)=\min \left\{\widehat{d}_{\lambda}(u): u \in W_{0}^{1, p(z)}(\Omega)\right\}, \\
\Rightarrow & \widetilde{u}_{0} \in K_{\widehat{d}_{\lambda}}=\left\{u_{0}\right\} \quad(\text { see }(54)) \\
\Rightarrow & \widetilde{u}_{0}=u_{0} \in \operatorname{int} C_{+}
\end{aligned}
$$

But then from (50) and (55), it follows that

$$
\begin{align*}
& u_{0} \in \operatorname{int} C_{+} \text {is a local } C_{0}^{1}(\bar{\Omega}) \text {-minimizer of } d_{\lambda}(\cdot), \\
& \Rightarrow u_{0} \in \operatorname{int} C_{+} \text {is a local } W_{0}^{1, p(z)}(\Omega) \text {-minimizer of } d_{\lambda}(\cdot)  \tag{56}\\
& \text { (see Proposition } A 3 \text { of the Appendix) } .
\end{align*}
$$

From (54), (56) and Proposition 5.7.6 of Papageorgiou, Rădulescu and Repovš [42, p. 449], we know that we can find $\rho \in(0,1)$ small such that

$$
\begin{equation*}
d_{\lambda}\left(u_{0}\right)<\inf \left\{d_{\lambda}(u):\left\|u-u_{0}\right\|=\rho\right\}=m_{\lambda} \tag{57}
\end{equation*}
$$

Given $u \in \operatorname{int} C_{+}$, on account of hypothesis $H_{1}$ (ii), we have

$$
\begin{equation*}
d_{\lambda}(t u) \rightarrow-\infty \text { as } t \rightarrow+\infty \tag{58}
\end{equation*}
$$

Also, reasoning as in the "Claim" in the proof of Proposition 4, we show that

$$
\begin{equation*}
d_{\lambda}(\cdot) \text { satisfies the } C \text {-condition. } \tag{59}
\end{equation*}
$$

Then (57), (58), (59) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W_{0}^{1, p(z)}(\Omega)$ such that

$$
\begin{equation*}
\widehat{u} \in K_{d_{\lambda}} \subseteq\left[u_{\mu}\right) \cap \operatorname{int} C_{+} \quad(\text { see }(53)) \text { and } m_{\lambda} \leq d_{\lambda}(\widehat{u}) \quad(\text { see }(57)) \tag{60}
\end{equation*}
$$

From (60), (57) and (51) it follows that

$$
\widehat{u} \in S_{\lambda} \subseteq \operatorname{int} C_{+}, \widehat{u} \neq u_{0}
$$

The proof is now complete.
To complete the proof of Theorem 2.1, it remains to show that the critical parameter value $\lambda^{*}$ is admissible (that is, $\lambda^{*} \in \mathfrak{L}$ ). This is done in the next proposition.

Proposition 13. If hypotheses $H_{0}, H_{1}$ hold, then $\lambda^{*} \in \mathfrak{L}$.
Proof. Let $\left\{\lambda_{n}\right\}_{n \geq 1} \subseteq \mathfrak{L}$ such that $\lambda_{n} \uparrow \lambda^{*}$. Also let $\mu \in\left(0, \lambda_{1}\right)$. We know that $\mu \in \mathfrak{L}$ and so we can find $u_{\mu} \in S_{\mu} \subseteq \operatorname{int} C_{+}$. Using $u_{\mu}(z)$, we can define $i_{\lambda_{n}}(z, x)$ (see (51)) and then introduce $d_{\lambda_{n}}(\cdot) \in C^{1}\left(W_{0}^{1, p(z)}(\Omega)\right)$ (see the proof of Proposition 12) for every $n \in \mathbb{N}$. From the proof of Proposition 12, we know that for every $n \in \mathbb{N}$, we can find $u_{n} \in S_{\lambda_{n}} \subseteq \operatorname{int} C_{+}$such that

$$
\begin{align*}
d_{\lambda_{n}}\left(u_{n}\right) & \leq d_{\lambda_{n}}\left(u_{\mu}\right) \quad\left(\text { see }(55) \text { and recall that } u_{\lambda_{n}} \text { is a minimizer of } \widehat{d}_{\lambda_{n}}\right) \\
& \leq \frac{1}{q_{-}}\left[\rho_{p}\left(D u_{\mu}\right)+\rho_{q}\left(D u_{\mu}\right)\right] \\
& -\int_{\Omega}\left[\lambda_{n} u_{\mu}^{-\eta(z)}+f\left(z, u_{\mu}\right)\right] u_{\mu} d z \quad(\text { see }(51)) . \tag{61}
\end{align*}
$$

We have

$$
\begin{align*}
& \int_{\Omega}\left[\lambda_{n} u_{\mu}^{-\eta(z)}+f\left(z, u_{\mu}\right)\right] u_{\mu} d z \\
\geq & \int_{\Omega}\left[\mu u_{\mu}^{-\eta(z)}+f\left(z, u_{\mu}\right)\right] u_{\mu} d z \quad\left(\text { see } \mu<\lambda_{1} \leq \lambda_{n} \text { for all } n \in \mathbb{N}\right) \\
= & \rho_{p}\left(D u_{\mu}\right)+\rho_{q}\left(D u_{\mu}\right) \quad\left(\text { since } u_{\mu} \in S_{\mu}\right) . \tag{62}
\end{align*}
$$

We return to (61) and use (62) to obtain

$$
\begin{equation*}
d_{\lambda_{n}}\left(u_{n}\right) \leq\left(\frac{1}{q_{-}}-1\right) \rho_{p}\left(D u_{\mu}\right)+\left(\frac{1}{q_{-}}-1\right) \rho_{q}\left(D u_{\mu}\right)<0 \tag{63}
\end{equation*}
$$

for all $n \in \mathbb{N}$.
Moreover, for every $n \in \mathbb{N}$, we have

$$
\begin{equation*}
\left\langle A_{p(z)}\left(u_{n}\right), g\right\rangle+\left\langle A_{q(z)}\left(u_{n}\right), g\right\rangle=\int_{\Omega} i_{\lambda_{n}}\left(z, u_{n}\right) g d z \tag{64}
\end{equation*}
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$.
From (63) and (64) and reasoning as in the "Claim" in the proof of Proposition 4, we show that

$$
\begin{equation*}
u_{n} \rightarrow u^{*} \text { in } W_{0}^{1, p(z)}(\Omega) \text { as } n \rightarrow \infty \tag{65}
\end{equation*}
$$

Recall that $u_{\mu} \leq u_{n}$ for all that $n \in \mathbb{N}$ (see (53)). Therefore

$$
\begin{equation*}
u_{\mu} \leq u^{*} \tag{66}
\end{equation*}
$$

Passing to the limit as $n \rightarrow \infty$ in (64) and using (65) and (66), we conclude that $u^{*} \in S_{\lambda^{*}}$, hence $\lambda^{*} \in \mathfrak{L}$. The proof of Proposition 13 is now complete.

So, we have proved that $\mathfrak{L}=\left(0, \lambda^{*}\right]$ and this completes the proof of Theorem 2.1.
7. Appendix. In this appendix we prove some results about general anisotropic boundary value problems, which are of independent interest. Our results developed here will be useful to everyone working on anisotropic double phase problems. The results established in this Appendix complement and improve several related results in the literature.
7.1. A boundedness property in the anisotropic singular case. We establish in what follows a boundedness result for the weak solutions of anisotropic singular Dirichlet problems. We know that such a result is the essential first step to have global $C^{1}$-regularity of the solutions (see Fukagai and Narukawa [17, Lemma 3.3] and Lieberman [33] for isotropic problems ). Next, we establish a strong maximum principle for problems driven by the anisotropic $(p, q)$-Laplacian (double phase problems). In the previous sections we have remarked how important is to know that the positive solutions belong in int $C_{+}$. A remarkable outgrowth of the previous two results is Proposition $A 3$, which relates local Hölder and Sobolev minimizers of a $C^{1}$-functional. This proposition led to the multiplicity property established in Proposition 12. Finally, we state in Proposition $A 4$ a strong comparison principle for anisotropic problems.

We start by showing that every weak solution of a general anisotropic singular problem is essentially bounded. This property extends to the anisotropic singular framework results obtained by Fan and Zhao [16], Giacomoni, Schindler and Takač [25], Byun and Ko [11].

So, we consider the following anisotropic Dirichlet problem

$$
-\Delta_{p(z)} u(z)-\Delta_{q(z)} u(z)=f(z, u(z)) \quad \text { in } \Omega,\left.u\right|_{\partial \Omega}=0, u \geq 0
$$

The hypotheses on the exponents remain the same (see hypotheses $H_{0}$ ), while the hypotheses on the reaction $f(z, x)$ are the following:
$H_{1}^{A}: f: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that

$$
|f(z, x)| \leq \widehat{c}_{1}\left[|x|^{-\eta(z)}+|x|^{r(z)-1}+1\right]
$$

with $r \in C(\bar{\Omega}), 1<r(z)<p^{*}(z)$ for all $z \in \bar{\Omega}$ and some $\widehat{c}_{1}>0$.
As before, by a "(weak) solution" of problem $\left(a_{1}\right)$, we mean a function $u \in$ $W_{0}^{1, p(z)}(\Omega)$ such that $u(z) \geq 0$ for a.a. $z \in \Omega, f(\cdot, u(\cdot)) g(\cdot) \in L^{1}(\Omega)$ for all $g \in$ $W_{0}^{1, p(z)}(\Omega)$ and

$$
\left\langle A_{p(z)}(u), g\right\rangle+\left\langle A_{q(z)}(u), g\right\rangle=\int_{\Omega} f(z, u) g d z
$$

for all $g \in W_{0}^{1, p(z)}(\Omega)$.
Our approach is based on the Moser iteration technique. In fact, we present two proofs based on this technique. This technique was used in the past in the context of isotropic problems; see Winkert [56], Hu and Papageorgiou [30], Marino and Winkert $[36,37]$. An alternative approach, can be based on estimates of the Ladyzhenskaya-Uraltseva type (see [31]). This method was used by Papageorgiou and Rădulescu [39] (for problems driven by a general isotropic nonhomogeneous differential operator) and by Byun and Ko [11] (anisotropic problems driven by the $p(z)$-Laplacian with singular terms). We mention also the work of Acerbi and Mingione [1], who obtained local estimates for the gradient $D u(\cdot)$ (Calderon-Zygmund type estimates).

Proposition A1. If hypotheses $H_{0}, H_{1}^{A}$ hold and $u \in W_{0}^{1, p(z)}(\Omega)$ is a weak solution of problem $\left(a_{1}\right)$, then $u \in L^{\infty}(\Omega)$.

Proof. Let $\xi: \mathbb{R} \mapsto[0,1]$ be a $C^{1}$-cutoff function which satisfies

$$
\begin{equation*}
\operatorname{supp} \xi \subseteq \mathbb{R}_{+},\left.\xi\right|_{[1,+\infty)} \equiv 1 \text { and } \xi^{\prime}(t) \geq 1 \text { for all } t \in[0,1] \tag{2}
\end{equation*}
$$

For every $\varepsilon>0$, we set

$$
\begin{equation*}
\xi_{\varepsilon}(t)=\xi\left(\frac{t-1}{\varepsilon}\right) \tag{3}
\end{equation*}
$$

From the chain rule for Sobolev functions (see Proposition 1.4.2 of Papageorgiou, Rădulescu and Repovš [42, p. 22]) and since $W_{0}^{1, p(z)}(\Omega) \hookrightarrow W_{0}^{1, p_{-}}(\Omega)$ continuously, we have

$$
\xi_{\varepsilon}(u) \in W_{0}^{1, p(z)}(\Omega), D \xi_{\varepsilon}(u)=\xi_{\varepsilon}^{\prime}(u) D u\left(\operatorname{see}\left(a_{3}\right)\right)
$$

Let $y \in W_{0}^{1, p(z)}(\Omega), y \geq 0$ and use as a test function $g=\xi_{\varepsilon}(u) y$. We have

$$
\begin{align*}
\left\langle A_{p(z)}(u), g\right\rangle & =\int_{\Omega}|D u|^{p(z)-2}\left(D u, D\left(\xi_{\varepsilon}(u) y\right)\right)_{\mathbb{R}^{N}} d z \\
& =\int_{\Omega}|D u|^{p(z)} \xi_{\varepsilon}^{\prime}(u) y d z+\int_{\Omega}|D u|^{p(z)-2}(D u, D y)_{\mathbb{R}^{N}} \xi_{\varepsilon}(u) d z \\
& \geq \int_{\Omega}|D u|^{p(z)-2}(D u, D y)_{\mathbb{R}^{N}} \xi_{\varepsilon}(u) d z \quad\left(\text { see }\left(a_{2}\right)\right) \tag{4}
\end{align*}
$$

Similarly, we show that

$$
\begin{align*}
\left\langle A_{q(z)}(u), g\right\rangle & =\int_{\Omega}|D u|^{q(z)-2}\left(D u, D\left(\xi_{\varepsilon}(u) y\right)\right)_{\mathbb{R}^{N}} d z \\
& \geq \int_{\Omega}|D u|^{q(z)-2}(D u, D y)_{\mathbb{R}^{N}} \xi_{\varepsilon}(u) d z \quad\left(\text { see }\left(a_{5}\right)\right) \tag{5}
\end{align*}
$$

Since $u \in W_{0}^{1, p(z)}(\Omega)$ is a weak solution of problem $\left(a_{1}\right)$, we have

$$
\begin{align*}
& \int_{\Omega}|D u|^{p(z)-2}(D u, D y)_{\mathbb{R}^{N}} \xi_{\varepsilon}(u) d z+\int_{\Omega}|D u|^{q(z)-2}(D u, D y)_{\mathbb{R}^{N}} \xi_{\varepsilon}(u) d z \\
\leq & \left\langle A_{p(z)}(u), g\right\rangle+\left\langle A_{q(z)}(u), g\right\rangle \quad\left(\text { see }\left(a_{4}\right),\left(a_{5}\right)\right) \\
= & \int_{\Omega} f(z, u) \xi_{\varepsilon}(u) y d z \\
\leq & \int_{\Omega} \widehat{c_{1}}\left[u^{-\eta(z)}+u^{r(z)-1}+1\right] \xi_{\varepsilon}(u) y d z \tag{6}
\end{align*}
$$

for all $\varepsilon \in(0,1]$.
Let $\Omega_{+}^{1}=\{z \in \Omega: u(z)>1\}$. We let $\varepsilon \rightarrow 0^{+}$in $\left(a_{6}\right)$ and obtain

$$
\begin{align*}
& \int_{\Omega_{+}^{1}}|D u|^{p(z)-2}(D u, D y)_{\mathbb{R}^{N}} d z+\int_{\Omega_{+}^{1}}|D u|^{q(z)-2}(D u, D y)_{\mathbb{R}^{N}} d z \\
\leq & \int_{\Omega_{+}^{1}} \widehat{c}_{2}\left[u^{r(z)-1}+1\right] y d z \tag{7}
\end{align*}
$$

for some $\widehat{c}_{2}>0$.
For $\theta \geq 1$, let $u_{\theta}=\min \{u, \theta\} \in W_{0}^{1, p(z)}(\Omega)$. Also let $\lambda>0$ and use as test function $y=u u_{\theta}{ }^{p_{+}}$. We obtain

$$
\begin{aligned}
& \int_{\Omega_{+}^{1}}|D u|^{p(z)-2}\left(D u, D\left(u u_{\theta}^{\lambda p_{+}}\right)\right)_{\mathbb{R}^{N}} d z \\
= & \int_{\Omega_{+}^{1}}|D u|^{p(z)} u_{\theta}{ }^{\lambda p_{+}} d z+\lambda p_{+} \int_{\Omega_{+}^{1} \cap\{u \leq \theta\}}|D u|^{p(z)} u_{\theta}^{\lambda p_{+}} d z \\
\geq & \frac{1}{2} \int_{\Omega_{+}^{1} \cap\{u>\theta\}}|D u|^{p(z)} u_{\theta}{ }^{\lambda p_{+}} d z+\left(\lambda p_{+}+1\right) \int_{\Omega_{+}^{1} \cap\{u \leq \theta\}}|D u|^{p(z)} u_{\theta}{ }^{\lambda p_{+}} d z \\
\geq & \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \int_{\Omega_{+}^{1} \cap\{u>\theta\}}|D u|^{p(z)} u_{\theta} p_{+}^{\lambda p_{+}} d z+\frac{\lambda p_{+}+1}{2} \int_{\Omega_{+}^{1} \cap\{u \leq \theta\}}|D u|^{p(z)} u_{\theta}{ }^{\lambda p_{+}} d z
\end{aligned}
$$

(using Bernoulli's inequality)
$\geq \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \int_{\Omega_{+}^{1}}|D u|^{p(z)} u_{\theta}{ }^{\lambda p_{+}} d z$
$\geq \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \int_{\Omega_{+}^{1}}\left|D\left(u u_{\theta}{ }^{\lambda}\right)\right|^{p(z)} d z \quad($ since $\theta \geq 1)$.
Similarly we have

$$
\begin{aligned}
& \int_{\Omega_{+}^{1}}|D u|^{q(z)-2}\left(D u, D\left(u u_{\theta}{ }^{\lambda p_{+}}\right)\right)_{\mathbb{R}^{N}} d z \\
\geq & \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \int_{\Omega_{+}^{1}}|D u|^{q(z)} u_{\theta}{ }^{\lambda p_{+}} d z \geq 0 .
\end{aligned}
$$

So, we can write

$$
\begin{aligned}
& \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \int_{\Omega_{+}^{1}}\left|D\left(u u_{\theta}^{\lambda}\right)\right|^{p(z)} d z \\
\leq & \widehat{c}_{2}\left[\int_{\Omega_{+}^{1}} u u_{\theta}{ }^{\lambda p_{+}} d z+\int_{\Omega_{+}^{1}} u^{r(z)} u_{\theta}{ }^{\lambda p_{+}} d z\right] \quad\left(\text { see }\left(a_{7}\right)\right) \\
\leq & \widehat{c}_{3} \int_{\Omega_{+}^{1}} u^{r_{+}} u_{\theta}{ }^{\lambda p_{+}} d z
\end{aligned}
$$

$$
\left(a_{8}\right)
$$

for some $\widehat{c}_{3}>0$.
Without any loss of generality we may assume that $\left\|u u_{\theta}{ }^{\lambda}\right\| \geq 1$. Then we have

$$
\begin{align*}
& \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \int_{\Omega_{+}^{1}}\left|D\left(u u_{\theta}^{\lambda}\right)\right|^{p(z)} d z \\
= & \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}} \rho_{p}^{\Omega_{+}^{1}}\left(D\left(u u_{\theta}^{\lambda}\right)\right) \\
\geq & \frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}}\left\|D\left(u u_{\theta}^{\lambda}\right)\right\|_{p(z), \Omega_{+}^{1}}^{p_{-}} \\
\geq & \widehat{c}_{4} \frac{\lambda p_{+}+1}{(\lambda+1)^{p_{+}}}\left\|u u_{\theta}{ }^{\lambda}\right\|_{L^{p_{+}}\left(\Omega_{+}^{1}\right)}^{p_{-}} \tag{9}
\end{align*}
$$

for some $\widehat{c}_{4}>0$ (using the anisotropic Sobolev embedding theorem).
We return to $\left(a_{8}\right)$ and use $\left(a_{9}\right)$. We obtain

$$
\begin{equation*}
\left\|u u_{\theta}^{\lambda}\right\|_{L^{p_{+}}\left(\Omega_{+}^{1}\right)}^{p_{-}} \leq \widehat{c}_{5} \int_{\Omega_{+}^{1}} u^{p_{+}^{*}} u_{\theta}^{\lambda p_{+}} d z \tag{10}
\end{equation*}
$$

for some $\widehat{c}_{5}>0$ (recall that $r_{+}<p_{+}^{*}$ ).
We choose $\lambda_{1}>0$ so that $\left(\lambda_{1}+1\right) p_{+}=p_{+}^{*}$. Then

$$
\left\|u u_{\theta}^{\lambda_{1}}\right\|_{L^{p_{+}}\left(\Omega_{+}^{1}\right)} \leq \widehat{c}_{6}\|u\|_{p_{+}^{*}}^{p_{+}^{*} / p_{-}}\left(\operatorname{see}\left(a_{10}\right)\right)
$$

for some $\widehat{c}_{6}=\widehat{c}_{6}\left(\lambda_{1}, u\right)>0$.
We let $\theta \rightarrow+\infty$ and use Fatou's lemma to obtain

$$
\|u\|_{L^{\left(\lambda_{1}+1\right) p_{+}}\left(\Omega_{+}^{1}\right)} \leq \widehat{c}_{6}\|u\|_{p_{+}^{*}}^{p_{+}^{*} / p_{-}} .
$$

Now we perform a typical bootstrap procedure. We repeat the above steps with $\lambda_{k}>0, k \in \mathbb{N}, k \geq 2$ such that

$$
\left(\lambda_{k}+1\right) p_{+}=\left(\lambda_{k-1}+1\right) p_{+}^{*} .
$$

Then for every $\lambda>0$, we have

$$
\begin{aligned}
& \|u\|_{L^{(\lambda+1) p_{+}\left(\Omega_{+}^{1}\right)}} \leq \widehat{c}_{7} \quad \text { for some } \widehat{c}_{7}=\widehat{c}_{7}(\lambda, u)>0, \\
\Rightarrow & u \in L^{\mu}\left(\Omega_{+}^{1}\right) \text { for every } \mu \in[1,+\infty) .
\end{aligned}
$$

From ( $a_{8}$ ) we have

$$
\frac{\lambda p_{+}+1}{2(\lambda+1)^{p_{+}}}\left\|D\left(u u_{\theta}^{\lambda}\right)\right\|_{L^{p_{+}}\left(\Omega_{+}^{1}\right)}^{p_{-}} \leq \widehat{c}_{3} \int_{\Omega_{+}^{1}} u^{p_{+}^{*}} u_{\theta}{ }^{\lambda p_{+}} d z
$$

Using Hölder's inequality, Fatou's lemma (as $\theta \rightarrow+\infty$ ) and the classical Moser iteration process (see, for example, Gasiński and Papageorgiou [20, pp. 333-334] and Marino and Winkert [36, pp. 164-165]), we obtain

$$
\|u\|_{L^{\left(\lambda_{k}+1\right) p_{+}^{*}}} \leq \widehat{c_{8}}
$$

for some $\widehat{c}_{8}>0$, all $k \in \mathbb{N}$, with $\lambda_{k} \rightarrow+\infty$.
Then from Problem 3.104 of Gasiński and Papageorgiou [22, p. 477] we infer that

$$
\begin{aligned}
& u \in L^{\infty}\left(\Omega_{+}^{1}\right), \\
\Rightarrow & u \in L^{\infty}(\Omega) .
\end{aligned}
$$

The proof is now complete.
Remark 2. The result can be easily extended also to anisotropic Neumann and Robin problems. Also, we can go beyond the anisotropic two-phase equations. Namely, we can replace the differential operator $u \mapsto-\Delta_{p(z)} u-\Delta_{q(z)} u$ with a more general operator of the form $u \mapsto-\operatorname{div} a(z, D u)$. Indeed, a careful inspection of the proof of Proposition $A 1$, reveals that the result remains true if we have the differential operator $u \mapsto-\operatorname{div} a(z, D u)$, with $a: \Omega \times \mathbb{R}^{N} \mapsto \mathbb{R}^{N}$ being a Carathéodory function satisfying

- $|a(z, y)| \leq \widehat{c}_{9}\left[1+|y|^{p(z)-1}\right]$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^{N}$, for some $\widehat{c}_{9}>0$;
- $(a(z, y), y)_{\mathbb{R}^{N}} \geq \widehat{c}_{10}|y|^{p(z)}$ for a.a. $z \in \Omega$, all $y \in \mathbb{R}^{N}$, for some $\widehat{c}_{10}>0$.

These hypotheses cover the case of the anisotropic $(p, q)$-Laplacian.
For the benefit of the reader, we outline an alternative proof of Proposition $A 1$. The argument in this proof can be used in other situations, too. We employ again the Moser iteration process, but now this is done in a different fashion, convenient for the anisotropic $(p, q)$-Laplacian.
7.1.1. Alternative proof of Proposition A1. The domain $\Omega \subseteq \mathbb{R}^{N}$ is relatively compact, thus totally bounded. So, given any $\rho>0$, we can find a finite number of $\rho$-balls $\left\{B_{k}(\rho)\right\}_{k=1}^{m}$ with centres in $\bar{\Omega}$ such that $\bar{\Omega} \subseteq \bigcup_{k=1}^{m} B_{k}(\rho)$. In what follows, we write for simplicity $B_{k}=B_{k}(\rho)$. We set

$$
p_{-}^{k}=\min _{\bar{\Omega} \cap \bar{B}_{k}} p(\cdot), p_{+}^{k}=\max _{\bar{\Omega} \cap \bar{B}_{k}} p(\cdot), r_{+}^{k}=\max _{\bar{\Omega} \cap \bar{B}_{k}} r(\cdot)
$$

Consider a smooth partition of unity $\left\{\psi_{k}\right\}_{k=1}^{m}$ subordinated to the open cover $\left\{B_{k}\right\}_{k=1}^{m}$. So, we have

$$
\operatorname{supp} \psi_{k} \subseteq B_{k}, 0 \leq \psi_{k} \leq 1, \sum_{k=1}^{m} \psi_{k}(z)=1 \text { for all } z \in \bar{\Omega}
$$

As before for every $\theta \geq 1$, let $u_{\theta}=\min \{u, \theta\} \in W_{0}^{1, p(z)}(\Omega)$. Let $\lambda>0$ and in the definition of weak solution use as test function $g=u_{\theta}{ }^{\lambda+1}$. We have

$$
\begin{align*}
& (\lambda+1)\left[\int_{\Omega}\left|D u_{\theta}\right|^{p(z)} u_{\theta}^{\lambda} d z+\int_{\Omega}\left|D u_{\theta}\right|^{q(z)} u_{\theta}^{\lambda} d z\right] \\
= & \int_{\Omega} f(z, u) u_{\theta}{ }^{\lambda+1} d z . \tag{12}
\end{align*}
$$

We have

$$
\begin{align*}
& (\lambda+1) \int_{\Omega}\left|D u_{\theta}\right|^{p(z)} u_{\theta}{ }^{\lambda} d z \\
= & (\lambda+1) \int_{\Omega}\left|D u_{\theta}\right|^{p(z)} u_{\theta}{ }^{\lambda}\left(\sum_{k=1}^{m} \psi_{k}\right) d z \quad\left(\operatorname{see}\left(a_{11}\right)\right) \\
\geq & (\lambda+1) \sum_{k=1}^{m} \int_{\Omega}\left|D u_{\theta}\right|^{p_{-}^{k}} u_{\theta}{ }^{\lambda} \psi_{k} d z-(\lambda+1) \sum_{k=1}^{m} \int_{\Omega} u_{\theta}{ }^{\lambda} \psi_{k} d z \\
\geq & \frac{\lambda+1}{\left(\frac{\lambda}{p_{-}}+1\right)^{p_{+}}} \sum_{k=1}^{m} \int_{\Omega}\left|D u_{\theta}\left(\lambda / p_{-}^{k}+1\right)\right|^{p_{-}^{k}} \psi_{k} d z \\
- & (\lambda+1) \widehat{c}_{11} \sum_{k=1}^{m}\left[\int_{\Omega} u^{\left(\lambda / p_{-}^{k}+1\right)} d z\right] \frac{p_{-\lambda}^{k}}{\left(\lambda+p_{-}^{k}\right) r_{+}^{k}} \\
\geq & \frac{\lambda+1}{\left(\frac{\lambda}{p_{-}}+1\right)^{p_{+}}} \sum_{k=1}^{m} \int_{\Omega}\left|D u_{\theta}\left(\lambda / p_{-}^{k}+1\right)\right|^{p_{-}^{k}} \psi_{k} d z \\
- & (\lambda+1) \widehat{c}_{12}\left[\sum_{k=1}^{m}\left\|u\left(\lambda / p_{-}^{k}+1\right)\right\|_{L_{-}^{r_{+}^{k}}(\Omega)}^{p^{k}}+1\right] \tag{13}
\end{align*}
$$

for some $\widehat{c}_{11}, \widehat{c}_{12}>0$.
Also we have

$$
\begin{equation*}
(\lambda+1) \int_{\Omega}\left|D u_{\theta}\right|^{q(z)} u_{\theta}^{\lambda} d z \geq 0 \tag{14}
\end{equation*}
$$

Moreover, using hypothesis $H_{1}^{A}$ we have

$$
\begin{aligned}
& \int_{\Omega} f(z, u) u_{\theta}{ }^{\lambda+1} d z \\
\leq & \widehat{c}_{1}\left[u_{\theta}{ }^{-\eta(z)}+u_{\theta}^{r(z)-1}+1\right] u_{\theta}{ }^{\lambda+1} d z \\
\leq & \left.\widehat{c}_{13}\left[1+u^{r(z)+\lambda}\right]\left(\sum_{k=1}^{m} \psi_{k}\right) d z \quad \text { (see hypotheses } H_{0}\right) \\
\leq & \widehat{c}_{14}\left[\sum_{k=1}^{m} \int_{\Omega} u^{\left(\lambda / p_{-}^{k}+1\right) p_{-}^{k}} u^{r_{+}^{k}-p_{-}^{k}} d z+1\right] \\
\leq & \widehat{c}_{15}\left[\sum_{k=1}^{m}\left\|u^{\left(\lambda / p_{-}^{k}+1\right)}\right\|_{L_{-}^{r}{ }_{+}^{k}(\Omega)}^{p_{+}^{k}}+1\right] \quad \text { (use Hölder's inequality), }
\end{aligned}
$$

for some $\widehat{c}_{13}, \widehat{c}_{14}, \widehat{c}_{15}>0$.
We return to $\left(a_{12}\right)$ and use $\left(a_{13}\right),\left(a_{14}\right),\left(a_{15}\right)$. We obtain

$$
\begin{aligned}
& \frac{\lambda+1}{\left(\frac{\lambda}{p_{-}}+1\right)^{p_{+}}} \sum_{k=1}^{m} \int_{\Omega}\left|D u_{\theta}\left(\lambda / p_{-}^{k}+1\right)\right|^{p_{-}^{k}} \psi_{k} d z \\
\leq & \widehat{c}_{16}(\lambda+1)\left[\sum_{k=1}^{m}\left\|u^{\left(\lambda / p_{-}^{k}+1\right)}\right\|_{L^{r_{+}^{k}}(\Omega)}^{p_{-}^{k}}+1\right]
\end{aligned}
$$

for some $\widehat{c}_{16}>0$.

We let $\theta \rightarrow+\infty$ and use Fatou's lemma. It follows that

$$
\sum_{k=1}^{m} \int_{\Omega}\left|D u^{\left(\lambda / p_{-}^{k}+1\right)}\right|^{p_{-}^{k}} \psi_{k} d z \leq \widehat{c}_{16}\left(\frac{1}{p_{-}}+1\right)^{p_{+}}\left[\sum_{k=1}^{m}\left\|u^{\left(\lambda / p_{-}^{k}+1\right)}\right\|_{L^{r_{+}^{k}(\Omega)}}^{p_{-}^{k}}+1\right] .
$$

From this relation, via the anisotropic Sobolev embedding theorem, we obtain

$$
\|u\|_{L}\left(\lambda / p_{-}^{k}+1\right) p_{-\left(\Omega_{k}\right)}^{k} \leq \widehat{c}_{17}\left[\|u\|_{L}\left(\lambda / p_{-}^{k}+1\right) r_{-\left(\Omega_{k}\right)}^{k}+1\right]
$$

for some $\widehat{c}_{17}>0$ and with $\Omega_{k}=\left\{\psi_{k}=1\right\}$.
Using this estimate, as in the previous proof, the Moser iteration process can be performed and leads to the conclusion that $u \in L^{\infty}(\Omega)$.
7.2. Strong maximum principle for anisotropic singular double phase problems. The main result of this subsection here extends Theorem 1.1 of Zhang [59], which does not cover the case of the anisotropic $(p, q)$-Laplacian (see conditions (5), (6) in [59]). A maximum principle for isotropic double phase problems was proved recently by Papageorgiou, Vetro and Vetro [47]. We point out that in both the aforementioned works, no singular term is involved. So, our result here generalizes their works in that direction, too.

Given $u \in W_{0}^{1, p(z)}(\Omega)$, we write that

$$
-\Delta_{p(z)} u-\Delta_{q(z)} u-\lambda u^{-\eta(z)} \geq 0 \quad \text { in } \Omega
$$

if and only if for all $g \in W_{0}^{1, p(z)}(\Omega)$ with $g(z) \geq 0$ for a.a. $z \in \Omega$ we have

$$
\int_{\Omega}|D u|^{p(z)-2}(D u, D g)_{\mathbb{R}^{N}} d z+\int_{\Omega}|D u|^{q(z)-2}(D u, D g)_{\mathbb{R}^{N}} d z \geq \lambda \int_{\Omega} u^{-\eta(z)} g d z
$$

Proposition $A 2$. If hypotheses $H_{0}$ hold and $u \in C_{+} \backslash\{0\}$ satisfies

$$
-\Delta_{p(z)} u-\Delta_{q(z)} u-\lambda u^{-\eta(z)} \geq 0 \quad \text { in } \Omega,
$$

then $u \in \operatorname{int} C_{+}$.
Proof. First we show that $u(z)>0$ for all $z \in \Omega$. Arguing by contradiction, suppose that we could find $z_{1}, z_{2} \in \Omega$ and an open ball $B_{2 \rho}\left(z_{2}\right) \subset \subset \Omega$ such that $z_{1} \in \partial B_{2 \rho}\left(z_{2}\right), u\left(z_{1}\right)=0$ and $\left.u\right|_{B_{2 \rho}\left(z_{2}\right)}>0$. So, $2 \rho=\left|z_{1}-z_{2}\right|$ and $\rho>0$ can be chosen small by fixing $z_{1}$ and letting $z_{2}$ to vary.

Let $m=\min \left\{u(z):\left|z-z_{2}\right|=\rho\right\}>0$. We see that

$$
m \rightarrow 0^{+} \text {and } \frac{m}{\rho} \rightarrow 0^{+} \text {as } \rho \rightarrow 0^{+} \quad \text { (L'Hopital's rule). }
$$

Also note that since $u \in C_{+} \backslash\{0\}$ and $u(z)=0$ with $z_{1} \in \Omega$, we have $D u\left(z_{1}\right)=0$. We consider the following annulus

$$
D=\left\{z \in \Omega: \rho<\left|z-z_{2}\right|<2 \rho\right\}
$$

We see that $\left.u\right|_{D}>0$. By hypotheses $H_{0}$, the exponents $p, q$ are Lipschitz continuous and so by Rademacher's theorem, they are differentiable for almost all $z \in \bar{\Omega}$ (see, for example, Theorem 5.8.12 of Papageorgiou and Winkert [48, p. 476]) So, we can define

$$
\theta=\sup \{\max \{|D p(z)|,|D q(z)|\}: z \in D\}<\infty
$$

We know that

$$
\frac{m}{\rho} \rightarrow 0^{+} \text {and } \frac{p(z)-1}{p\left(z_{1}\right)-1} \rightarrow 1 \text { uniformly on } D \text { as } \rho \rightarrow 0^{+} .
$$

So, we can choose $\rho \in(0,1)$ small such that

$$
\frac{m}{\rho}<1 \text { and } \frac{p(z)-1}{p\left(z_{1}\right)-1} \geq \frac{1}{2} \text { for all } z \in D .
$$

In what follows, $p_{1}=p\left(z_{1}\right)$. Also we set

$$
\tau=-k \theta \ln \left(\frac{m}{\rho}\right)+\frac{2(N-1)}{\rho}, k>2
$$

and consider the function

$$
v(t)=\frac{m\left(e^{\frac{\tau t}{p_{1}-1}}-1\right)}{e^{\frac{\tau t}{p_{1}-1}}-1} \text { for all } t \in[0, \rho] .
$$

Evidently, $v \in C^{2}[0, \rho]$ and $v^{\prime}, v^{\prime \prime}>0$ on $[0, \rho]$, hence they are both strictly increasing. In fact, we have

$$
\begin{equation*}
\frac{m}{\rho}<v^{\prime}(t)<1 \text { for all } t \in[0, \rho] \tag{17}
\end{equation*}
$$

For notational simplicity and without any loss of generality, we may assume that $z_{2}=0$. We denote

$$
s=\left|z-z_{2}\right|=|z| \text { and } t=2 \rho-s
$$

and introduce the function $w(s)$ defined by

$$
w(s)=v(2 \rho-s)=v(t) \text { for all } s \in[\rho, 2 \rho] .
$$

We have $w \in C^{2}[\rho, 2 \rho]$ and $w^{\prime}(s)=-v^{\prime}(t), w^{\prime \prime}(t)=v^{\prime \prime}(t)$. For every $z \in D$, $|z|=s$, we write $w(z)=w(s)$. Then $w \in C^{2}(D)$ and we have

$$
\begin{align*}
& \operatorname{div}\left[|D w|^{p(z)-2} D w+|D w|^{q(z)-2} D w\right] \\
= & {[p(z)-1] v^{\prime}(t)^{p(z)-1} v^{\prime \prime}(t)-\frac{N-1}{s} v^{\prime}(t)^{p(z)-1} } \\
- & v^{\prime}(t)^{p(z)-1} \ln v^{\prime}(t) \sum_{k=1}^{N} \frac{\partial p}{\partial z_{k}} \frac{z_{k}}{s} \\
+ & {[q(z)-1] v^{\prime}(t)^{q(z)-1} v^{\prime \prime}(t)-\frac{N-1}{s} v^{\prime}(t)^{q(z)-1} } \\
- & v^{\prime}(t)^{q(z)-1} \ln v^{\prime}(t) \sum_{k=1}^{N} \frac{\partial q}{\partial z_{k}} \frac{z_{k}}{s} . \tag{18}
\end{align*}
$$

Note that

$$
\left(p_{1}-1\right) v^{\prime \prime}(t)=\tau v^{\prime}(t)
$$

Using this equality in $\left(a_{18}\right)$, we obtain

$$
\begin{gathered}
\operatorname{div}\left[|D w|^{p(z)-2} D w+|D w|^{q(z)-2} D w\right] \\
> \\
v^{\prime}(t)^{p(z)-1}\left[\tau+2 \theta \ln \left(\frac{m}{\rho}\right)^{2}-\frac{2(N-1)}{s}\right]
\end{gathered}
$$

see $\left(a_{17}\right)$ and note that

$$
\left|\sum_{k-1}^{m} \frac{\partial p}{\partial z_{k}} \frac{z_{k}}{s}\right| \leq \theta, \quad\left|\sum_{k-1}^{m} \frac{\partial q}{\partial z_{k}} \frac{z_{k}}{s}\right| \leq \theta
$$

Then on account of $\left(a_{16}\right)$, we have

$$
\begin{aligned}
& \operatorname{div}\left[|D w|^{p(z)-2} D w+|D w|^{q(z)-2} D w\right]>0 \text { in } \Omega \\
\Rightarrow & -\Delta_{p(z)} u-\Delta_{q(z)} u-\lambda u^{-\eta(z)} \geq-\Delta_{p(z)} w-\Delta_{q(z)} w-\lambda w^{-\eta(z)} \text { in } D . \quad\left(a_{19}\right)
\end{aligned}
$$

Note that $v(0)=0, v(\rho)=m$. It follows that

$$
\begin{equation*}
\left.w\right|_{\partial D} \leq\left. u\right|_{\partial D} \tag{20}
\end{equation*}
$$

From $\left(a_{19}\right),\left(a_{20}\right)$ and the weak comparison principle, we obtain

$$
w(z) \leq u(z) \text { for all } z \in \bar{D}
$$

So, we have

$$
\frac{\partial u}{\partial n}\left(z_{1}\right) \leq \frac{\partial w}{\partial n}\left(z_{1}\right)=-v^{\prime}(0)<0 \quad\left(\text { recall that } u\left(z_{1}\right)=0\right)
$$

But from $\left(a_{21}\right)$ we infer that $D u\left(z_{1}\right) \neq 0$, a contradiction. Therefore we have proved that

$$
u(z)>0 \text { for all } z \in \Omega
$$

Finally, let $z_{1} \in \partial \Omega, B_{2 \rho}\left(z_{2}\right) \subseteq \Omega$. From the previous part of the proof, we know that we can find a function $w \in C^{2}(D) \cap C^{1}(\bar{D})$ such that

$$
w\left(z_{1}\right)=0, \frac{\partial w}{\partial n}\left(z_{1}\right)<0 \text { and }\left.w\right|_{\bar{D}} \leq\left. u\right|_{\bar{D}}
$$

It follows that $\frac{\partial u}{\partial n}\left(z_{1}\right)<0$, hence $u \in \operatorname{int} C_{+}$.
7.3. Anisotropic principle for local minimizers. In what follows, we present an anisotropic version of a well-known result of Brezis and Nirenberg [10], relating $C^{1}$ and Sobolev local minimizers of $C^{1}$-functionals. Since then the result has been extended in many different directions. We mention the works of Garcia Azorero, Manfredi and Peral Alonso [18] (Dirichlet problems driven by the isotropic p-Laplacian), Gasiński and Papageorgiou [21] (nonlinear Dirichlet problems with a nonsmooth potential), Gasiński and Papageorgiou [20] (Neumann problems driven by the anisotropic $p$-Laplacian) and Papageorgiou and Rădulescu [39] (Robin problems driven by an isotropic nonhomogeneous differential operator).

We consider a function $g: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ of the form

$$
g(z, x)=f(z)+g_{0}(z, x)
$$

with $f \in L^{s}(\Omega)$ for $s>N$ and $g_{0}: \Omega \times \mathbb{R} \mapsto \mathbb{R}$ a Carathéodory function satisfying

$$
\left|g_{0}(z, x)\right| \leq a(z)\left[1+|x|^{r(z)-1}\right] \text { for a.a. } z \in \Omega, \text { all } x \in \mathbb{R}
$$

with $a \in L^{\infty}(\Omega), r \in C(\bar{\Omega})$ and $1<r(z)<p^{*}(z)$ for all $z \in \bar{\Omega}$.
We set $G(z, x)=\int_{0}^{x} g(x, s) d s$ and consider the $C^{1}$-functional $k: W_{0}^{1, p(z)}(\Omega) \mapsto \mathbb{R}$ defined by

$$
k(u)=\int_{\Omega} \frac{1}{p(z)}|D u|^{p(z)} d z+\int_{\Omega} \frac{1}{q(z)}|D u|^{q(z)} d z-\int_{\Omega} G(z, u) d z
$$

for all $u \in W_{0}^{1, p(z)}(\Omega)$.
As before, we assume that the exponent $p(\cdot)$ and $q(\cdot)$ involved in the definition of $k(\cdot)$ satisfy the conditions in hypotheses $H_{0}$.

Proposition A3. If $u_{0} \in W_{0}^{1, p(z)}(\Omega)$ is a local $C_{0}^{1}(\bar{\Omega})$-minimizer of $k(\cdot)$, that is, there exists $\rho_{1}>0$ such that

$$
k\left(u_{0}\right) \leq k\left(u_{0}+h\right) \text { for all } h \in C_{0}^{1}(\bar{\Omega}),\|h\|_{C_{0}^{1}(\Omega)} \leq \rho_{1},
$$

then $u_{0} \in C_{0}^{1}(\bar{\Omega})$ and it is a local $W_{0}^{1, p(z)}(\Omega)$-minimizer of $k(\cdot)$, that is, there exists $\rho_{2}>0$ such that

$$
k\left(u_{0}\right) \leq k\left(u_{0}+h\right) \text { for all } h \in W_{0}^{1, p(z)}(\Omega),\|h\| \leq \rho_{2}
$$

Proof. Since, by hypothesis, $u_{0}$ is a local $C_{0}^{1}(\bar{\Omega})$-minimizer of $k(\cdot)$, for every $h \in$ $C_{0}^{1}(\bar{\Omega})$ and for $t \in(0,1)$ small, we have

$$
\begin{aligned}
& 0 \leq \frac{1}{t}\left[k\left(u_{0}+t h\right)-k\left(u_{0}\right)\right], \\
\Rightarrow & 0 \leq\left\langle k^{\prime}\left(u_{0}\right), h\right\rangle \text { for all } h \in C_{0}^{1}(\bar{\Omega}), \\
\Rightarrow & \left\langle k^{\prime}\left(u_{0}\right), h\right\rangle=0 \text { for all } h \in C_{0}^{1}(\bar{\Omega}) .
\end{aligned}
$$

The density of $C_{0}^{1}(\bar{\Omega})$ in $W_{0}^{1, p(z)}(\Omega)$ implies that

$$
\begin{aligned}
& k^{\prime}\left(u_{0}\right)=0 \text { in } W^{-1, p(z)}(\Omega)=W_{0}^{1, p(z)}(\Omega)^{*}, \\
\Rightarrow & \left\langle A_{p(z)}\left(u_{0}\right), h\right\rangle+\left\langle A_{q(z)}\left(u_{0}\right), h\right\rangle=\int_{\Omega} g\left(z, u_{0}\right) h d z \text { for all } h \in W_{0}^{1, p(z)}(\Omega), \\
\Rightarrow & -\Delta_{p(z)} u-\Delta_{q(z)} u=f(z)+g_{0}\left(z, u_{0}\right) \text { in } \Omega,\left.u_{0}\right|_{\partial \Omega}=0 .
\end{aligned}
$$

Since $f \in L^{s}(\Omega), s>N$, reasoning as in the proof of Proposition 7, we infer that

$$
u_{0} \in C_{0}^{1, \alpha}(\bar{\Omega})=C^{1, \alpha}(\bar{\Omega}) \cap C_{0}^{1}(\bar{\Omega}) \text { with } \alpha=1-\frac{N}{s} \in(0,1) .
$$

Let $\bar{B}_{\varepsilon}=\left\{u \in W_{0}^{1, p(z)}(\Omega):\|u\| \leq \varepsilon\right\}(\varepsilon>0)$. We set

$$
\begin{equation*}
m_{0}^{\varepsilon}=\inf \left\{k\left(u_{0}+h\right): h \in \bar{B}_{\varepsilon}\right\} . \tag{22}
\end{equation*}
$$

Arguing by contradiction, suppose that the second assertion of the proposition is not true. Then we can find $\varepsilon_{0}>0$ such that

$$
\begin{equation*}
m_{0}^{\varepsilon}<k\left(u_{0}\right) \text { for all } 0<\varepsilon \leq \varepsilon_{0} \tag{23}
\end{equation*}
$$

Let $\varepsilon \in\left(0, \varepsilon_{0}\right]$ and consider a sequence $\left\{v_{n}\right\}_{n \geq 1} \subseteq \bar{B}_{\varepsilon}$ such that

$$
\begin{equation*}
k\left(u_{0}+v_{n}\right) \downarrow m_{0}^{\varepsilon} \text { as } n \rightarrow \infty \tag{24}
\end{equation*}
$$

We may assume that

$$
v_{n} \xrightarrow{w} v_{\varepsilon} \text { in } W_{0}^{1, p(z)}(\Omega) \text { and } v_{n} \rightarrow v_{\varepsilon} \text { in } L^{r(z)}(\Omega) \text { as } n \rightarrow \infty
$$

The functional $k(\cdot)$ is sequentially weakly lower semicontinuous. So, from $\left(a_{25}\right)$ and $\left(a_{24}\right)$, we have

$$
\begin{aligned}
& k\left(u_{0}+v_{\varepsilon}\right) \leq m_{0}^{\varepsilon} \text { and } v_{\varepsilon} \in \bar{B}_{\varepsilon}, \\
\Rightarrow & k\left(u_{0}+v_{\varepsilon}\right)=m_{0}^{\varepsilon}<k\left(u_{0}\right) \quad\left(\text { see }\left(a_{22}\right),\left(a_{23}\right)\right), \\
\Rightarrow & v_{\varepsilon} \neq 0
\end{aligned}
$$

By the Lagrange multiplier rule (see Theorem 5.5.9 of Papageorgiou, Rădulescu and Repovš [42, p. 422]), we can find $\theta_{\varepsilon} \leq 0$ such that

$$
\begin{align*}
& k^{\prime}\left(u_{0}+v_{\varepsilon}\right)=\theta_{\varepsilon} A_{p(z)}\left(v_{\varepsilon}\right) \\
\Rightarrow & -\Delta_{p(z)}\left(u_{0}+v_{\varepsilon}\right)-\Delta_{q(z)}\left(u_{0}+v_{\varepsilon}\right)=g\left(z, u_{0}+v_{\varepsilon}\right)+\theta_{\varepsilon}\left|D v_{\varepsilon}\right|^{p(z)-2} D v_{\varepsilon} \text { in } \Omega \tag{26}
\end{align*}
$$

If $\theta_{\varepsilon}=0$, then from $\left(a_{26}\right)$ we infer as above that

$$
v_{\varepsilon} \in C_{0}^{1, \alpha}(\bar{\Omega}) \text { with } \alpha=1-\frac{N}{s} \in(0,1) \text { and }\left\|v_{\varepsilon}\right\|_{C_{0}^{1, \alpha}(\bar{\Omega})} \leq \widehat{c}_{18}
$$

for some $\widehat{c}_{18}>0$, all $\varepsilon \in\left(0, \varepsilon_{0}\right]$.
Next, suppose that $-1 \leq \theta_{\varepsilon}<0$. We know that

$$
\begin{equation*}
-\theta_{\varepsilon} \Delta_{p(z)} u_{0}-\theta_{\varepsilon} \Delta_{q(z)} u_{0}=\theta_{\varepsilon} g\left(z, u_{0}\right) \quad \text { in } \Omega \tag{28}
\end{equation*}
$$

We add $\left(a_{26}\right)$ and $\left(a_{28}\right)$ and obtain

$$
-\Delta_{p(z)}\left(u_{0}+v_{\varepsilon}\right)-\Delta_{q(z)}\left(u_{0}+v_{\varepsilon}\right)-\theta_{\varepsilon} \Delta_{p(z)} u_{0}-\theta_{\varepsilon} \Delta_{q(z)} u_{0}-\theta_{\varepsilon}\left|D v_{\varepsilon}\right|^{p(z)-2} D v_{\varepsilon}
$$

$$
\begin{equation*}
=g\left(z, u_{0}+v_{\varepsilon}\right)+\theta_{\varepsilon} g\left(z, u_{0}\right) \quad \text { in } \Omega \tag{29}
\end{equation*}
$$

We consider the map $a_{\varepsilon}: \Omega \times \mathbb{R}^{N} \mapsto \mathbb{R}^{N}$ defined by

$$
\begin{aligned}
a_{\varepsilon}(z, y) & =|y|^{p(z)-2} y+|y|^{q(z)-2} y+\theta_{\varepsilon}\left|y-D u_{0}(z)\right|^{q(z)-2}\left(y-D u_{0}(z)\right) \\
& +\theta_{\varepsilon}\left[\left|y-D v_{\varepsilon}(z)\right|^{p(z)-2}\left(y-D v_{\varepsilon}(z)+\left|y-D v_{\varepsilon}(z)\right|^{q(z)-2}\left(y-D v_{\varepsilon}(z)\right)\right]\right.
\end{aligned}
$$

Using $a_{\varepsilon}(z, y)$ we rewrite $\left(a_{29}\right)$ as follows

$$
-\operatorname{div} a_{\varepsilon}\left(z, D\left(u_{0}+v_{\varepsilon}\right)\right)=g\left(z, u_{0}+v_{\varepsilon}\right)+\theta_{\varepsilon} g\left(z, u_{0}\right) \quad \text { in } \Omega
$$

From Proposition $A 1$ (see also Theorem 4.1 of Fan and Zhao [16]) we have $u_{0}+$ $v_{\varepsilon} \in L^{\infty}(\Omega)$ and then by Lemma 3.3 of Fukagai and Narukawa [17], we have that ( $a_{27}$ ) holds.

Finally, suppose that $\theta_{\varepsilon}<-1$. In this case we have

$$
\begin{align*}
& -\Delta_{p(z)}\left(u_{0}+v_{\varepsilon}\right)-\Delta_{q(z)}\left(u_{0}+v_{\varepsilon}\right)+\Delta_{p(z)} u_{0}+\Delta_{q(z)} u_{0}-\theta_{\varepsilon}\left|D v_{\varepsilon}\right|^{p(z)-2} D v_{\varepsilon} \\
= & g\left(z, u_{0}+v_{\varepsilon}\right)-g\left(z, u_{0}\right) \quad \text { in } \Omega . \tag{30}
\end{align*}
$$

We introduce the map $\widehat{a}_{\varepsilon}: \Omega \times \mathbb{R}^{N} \mapsto \mathbb{R}^{N}$ defined by

$$
\begin{aligned}
\widehat{a}_{\varepsilon}(z, y) & =\frac{1}{\left|\theta_{\varepsilon}\right|}\left[\left|y+D u_{0}(z)\right|^{p(z)-2}\left(y+D u_{0}(z)\right)+\left|y+D u_{0}(z)\right|^{q(z)-2}\left(y+D u_{0}(z)\right)\right. \\
& \left.-\left|D u_{0}(z)\right|^{p(z)-2} D u_{0}(z)-\left|D u_{0}(z)\right|^{q(z)-2} D u_{0}(z)\right]+|y|^{p(z)-2} y
\end{aligned}
$$

Then using $\widehat{a}_{\varepsilon}(z, y)$, we rewrite $\left(a_{30}\right)$ as follows

$$
-\operatorname{div} \widehat{a}_{\varepsilon}\left(z, D u_{0}+D v_{\varepsilon}\right)=\frac{1}{\left|\theta_{\varepsilon}\right|}\left[g\left(z, u_{0}+v_{\varepsilon}\right)-g\left(z, u_{0}\right)\right] .
$$

As before, we infer that $\left(a_{27}\right)$ remains true.
So, in all cases, we have the validity of $\left(a_{27}\right)$. Let $\varepsilon_{n} \downarrow 0$ and let $v_{n}=v_{\varepsilon_{n}}$. From $\left(a_{27}\right)$ and the compact embedding of $C_{0}^{1, \alpha}(\bar{\Omega})$ into $C_{0}^{1}(\bar{\Omega})$, we see that by passing to a subsequence if necessary, we may assume that

$$
\begin{aligned}
& v_{n} \rightarrow v^{*} \text { in } C_{0}^{1}(\Omega) \\
\Rightarrow & \left.\left\|v^{*}\right\| \leq \varepsilon_{n} \text { for all } n \in \mathbb{N} \quad \text { (recall that } v_{n} \in \bar{B}_{\varepsilon_{n}}\right), \\
\Rightarrow & v^{*}=0
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
& v_{n} \rightarrow 0 \text { in } C_{0}^{1}(\bar{\Omega}) \\
\Rightarrow & \left\|v_{n}\right\|_{C_{0}^{1}(\bar{\Omega})} \leq \rho_{1} \text { for all } n \geq n_{0} \\
\Rightarrow & k\left(u_{0}\right) \leq k\left(u_{0}+v_{n}\right)=m_{0}^{\varepsilon_{n}}<k\left(u_{n}\right) \text { for all } n \geq n_{0} \quad(\text { see }(19)),
\end{aligned}
$$

a contradiction. This proves that $u_{0}$ is also a local $W_{0}^{1, p(z)}(\Omega)$-minimizer of $k(\cdot)$. The proof is now complete.

Remark 3. In fact, the proposition remains true for the more general functional

$$
\widehat{k}(u)=\int_{\Omega} \widehat{F}(D u) d z-\int_{\Omega} G(z, u) d z \text { for all } u \in W_{0}^{1, p(z)}(\Omega)
$$

with $\widehat{F}(y)$ a function such that

$$
\widehat{F}(y)=\widehat{F}_{0}(|y|) \text { for all } y \in \mathbb{R}^{N}
$$

with $\widehat{F}_{0}(t)=\int_{0}^{t} \widehat{f}_{0}(s) d s$, where $\widehat{f}_{0} \in C^{1}(0,+\infty)$ and there exists a function $\theta \in$ $C^{1}(0,+\infty)$ such that

$$
0<\widehat{c} \leq \frac{\theta^{\prime}(t) t}{\theta(t)} \leq \widehat{c}_{0} \quad \text { and } \quad \widehat{c}_{19} t^{p(z)-1} \leq \theta(t) \leq \widehat{c}_{20}\left[t^{p(z)-1}+t^{q(z)-1}\right]
$$

for some $\widehat{c}_{19}, \widehat{c}_{20}>0$. For $\widehat{f}(y)=\widehat{f}_{0}(|y|) y$, we have

$$
\begin{aligned}
& |\nabla \widehat{f}(y)| \leq \widehat{c}_{21} \frac{\theta(|y|)}{|y|} \text { for some } \widehat{c}_{21}>0, \text { all } y \in \mathbb{R}^{N}, \\
& (\nabla \widehat{f}(y) \xi, \xi)_{\mathbb{R}^{N}} \geq \frac{\theta(|y|)}{|y|}|\xi|^{2} \text { for all } y \in \mathbb{R}^{N} \backslash\{0\}, \text { all } \xi \in \mathbb{R}^{N}
\end{aligned}
$$

(see Fukagai and Narukawa [17] and Lieberman [33]).
Also, we mention that Proposition $A 3$ is also true for the energy functionals of Neumann and Robin problems.

Finally, we state a strong comparison principle. The proof at this result can be obtained by combining the proofs of Proposition 2.5 of Papageorgiou, Rădulescu and Repovš [43] and Proposition 6 of Papageorgiou, Rădulescu and Repovš [41]. We can state the following comparison result.

Proposition A4. If $u, v \in C^{1}(\bar{\Omega}), \widehat{\xi}, g_{1}, g_{2} \in L^{\infty}(\Omega), \widehat{\xi} \geq 0,0<\beta_{0} \leq g_{2}(z)-g_{1}(z)$ for a.a. $z \in \Omega, 0 \leq u \leq v$ in $\Omega$ and

$$
\begin{aligned}
& -\Delta_{p(z)} u-\Delta_{q(z)} u+\widehat{\xi}(z) u^{p(z)-1}-\lambda u^{-\eta(z)}=g_{1} \quad \text { in } \Omega \\
& -\Delta_{p(z)} v-\Delta_{q(z)} v+\widehat{\xi}(z) v^{p(z)-1}-\lambda v^{-\eta(z)}=g_{2} \quad \text { in } \Omega
\end{aligned}
$$

then $v-u \in D_{+}$.
The results of this Appendix provide all the basic tools for one to conduct an in-depth study of various anisotropic boundary value problems, singular and nonsingular alike.

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