Robin double-phase problems with singular and superlinear terms

Nikolaos S. Papageorgiou\textsuperscript{a}, Vicențiu D. Rădulescu\textsuperscript{b,c,}\*, Dušan D. Repovš\textsuperscript{d,e}

\textsuperscript{a} National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
\textsuperscript{b} Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland
\textsuperscript{c} Department of Mathematics, University of Craiova, 200585 Craiova, Romania
\textsuperscript{d} Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia
\textsuperscript{e} Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia

\textbf{ARTICLE INFO}

\textbf{Article history:}
Received 14 July 2020
Accepted 5 September 2020
Available online xxxx

\textbf{Keywords:}
Nonhomogeneous differential operator
Nonlinear regularity theory
Truncation
Strong comparison principle
Positive solutions

\textbf{ABSTRACT}

We consider a nonlinear Robin problem driven by the sum of $p$-Laplacian and $q$-Laplacian (i.e. the $(p,q)$-equation). In the reaction there are competing effects of a singular term and a parametric perturbation $\lambda f(z,u(z))$, which is Carathéodory and $(p-1)$-superlinear at $x \in \mathbb{R}$, without satisfying the Ambrosetti–Rabinowitz condition. Using variational tools, together with truncation and comparison techniques, we prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies.

© 2020 Elsevier Ltd. All rights reserved.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial \Omega$. In this paper, we study the following nonlinear Robin problem

\[
\begin{aligned}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z) u(z)^{p-1} &= u(z)^{-\gamma} + \lambda f(z,u(z)) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n_{pq}} + \beta(z) u^{p-1} &= 0 \quad \text{on } \partial \Omega, \quad u > 0, \quad \lambda > 0, \quad 0 < \gamma < 1, \quad 1 < q < p.
\end{aligned}
\]

\((P_\lambda)\)

For every $r \in (1,\infty)$, we denote by $\Delta_r$ the $r$-Laplace differential operator defined by

$\Delta_r u = \text{div}(|Du|^{r-2}Du)$ for all $u \in W^{1,r}(\Omega)$.

The differential operator of $(P_\lambda)$ is the sum of $p$-Laplacian and $q$-Laplacian. Such an operator is not homogeneous and it appears in the mathematical models of various physical processes. We mention the
works of Cherfils & Ilyasov [1] (reaction–diffusion systems) and Zhikov [2] (elasticity theory). The potential function \( \xi \in L^\infty(\Omega) \) satisfies \( \xi(z) \geq 0 \) for almost all \( z \in \Omega \). In the reaction (the right-hand side of (P\( _\lambda \)), we have the combined effects of two nonlinearities of different nature. One nonlinearity is the singular term \( u^{-\gamma} \) and the other nonlinearity is the parametric term \( \lambda f(z, x) \), where \( f(z, x) \) is a Carathéodory function (that is, for all \( x \in \mathbb{R} \), the mapping \( z \mapsto f(z, x) \) is measurable and for almost all \( z \in \Omega \), the mapping \( x \mapsto f(z, x) \) is continuous), which exhibits \((p-1)\)-superlinear growth near \(+\infty\) but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (the AR-condition for short). In the boundary condition, \( \frac{\partial u}{\partial n_{pq}} \) denotes the conormal derivative corresponding to the \((p, q)\)-Laplace differential operator. Then according to the nonlinear Green’s identity (see Gasinski & Papageorgiou [3, p. 210]), we have

\[
\frac{\partial u}{\partial n_{pq}} = (|Du|^{p-2}Du + |Du|^{q-2}Du, n) \quad \text{for all } u \in C^1(\overline{\Omega}),
\]

with \( n(\cdot) \) being the outward unit normal on \( \partial \Omega \). The boundary coefficient \( \beta \in C^{0,\alpha}(\partial \Omega) \) (with \( 0 < \alpha < 1 \)) satisfies \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \).

In the past, nonlinear singular problems were studied only in the context of Dirichlet equations driven by the \( p \)-Laplacian (a homogeneous differential operator). We mention the works of Giacomoni, Schindler & Takač [4], Papageorgiou, Rădulescu & Repovš [5,6], Papageorgiou & Smyrlis [7], Papageorgiou & Winkert [8], and Perera & Zhang [9]. Nonlinear elliptic problems with unbalanced growth have been studied recently by Papageorgiou, Rădulescu and Repovš [10–12]. Double-phase transonic flow problems with variable growth have been considered by Bahrouni, Rădulescu and Repovš [13]. A comprehensive study of semilinear singular problems can be found in the book of Ghergu & Rădulescu [14].

Using variational methods based on the critical point theory together with suitable truncation and comparison techniques, we prove a bifurcation type result, describing in a precise way the dependence of the set of positive solutions of \((P\_\lambda)\) on the parameter. So, we produce a critical parameter value \( \lambda^* > 0 \) such that for all \( \lambda \in (0, \lambda^*) \), problem \((P\_\lambda)\) has at least two positive solutions, for \( \lambda = \lambda^* \) problem \((P\_\lambda)\) has at least one positive solution and for \( \lambda > \lambda^* \) there are no positive solutions for problem \((P\_\lambda)\).

2. Mathematical background and hypotheses

Let \( X \) be a Banach space. By \( X^* \) we denote the topological dual of \( X \). Given \( \varphi \in C^1(X, \mathbb{R}) \), we say that \( \varphi(\cdot) \) satisfies the “C-condition”, if the following property holds

“Every sequence \( \{u_n\}_{n \geq 1} \subseteq X \) such that \( \{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R} \) is bounded and \((1 + \|u_n\|)\varphi'(u_n) \to 0 \) in \( X^* \) as \( n \to \infty \), admits a strongly convergent subsequence.”

This is a compactness type condition on the functional \( \varphi \), which leads to the minimax theory of the critical values of \( \varphi(\cdot) \).

The two main spaces in the analysis of problem \((P\_\lambda)\) are the Sobolev space \( W^{1,p}(\Omega) \) and the Banach space \( C^1(\overline{\Omega}) \). By \( \| \cdot \| \) we denote the norm of the Sobolev space \( W^{1,p}(\Omega) \). We have

\[
\|u\| = \left[ \|u\|^p_p + \|Du\|^{p}_{L^p} \right]^{\frac{1}{p}} \quad \text{for all } u \in W^{1,p}(\Omega).
\]

The Banach space \( C^1(\overline{\Omega}) \) is ordered with positive (order) cone given by

\[ C_+ = \{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \}. \]

This cone has a nonempty interior

\[ D_+ = \{ u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega} \}. \]
We will also consider another order cone (closed convex cone) in \( C^1(\overline{\Omega}) \), namely the cone

\[
\hat{C}_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}, \frac{\partial u}{\partial n}|_{\partial \Omega \cap u^{-1}(0)} \leq 0 \right\}.
\]

This cone has a nonempty interior

\[
\text{int} \hat{C}_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial \Omega \cap u^{-1}(0)} < 0 \right\}.
\]

To take care of the Robin boundary condition, we will also use the “boundary” Lebesgue spaces \( L^q(\partial \Omega)(1 \leq q \leq \infty) \). More precisely, on \( \partial \Omega \) we consider the \((N-1)\)-dimensional Hausdorff (surface) measure \( \sigma(\cdot) \). Using this measure on \( \partial \Omega \) we can define in the usual way the Lebesgue spaces \( L^q(\partial \Omega)(1 \leq q \leq \infty) \). We know that there exists a continuous, linear map \( \gamma_0 : W^{1,p}(\Omega) \to L^p(\partial \Omega) \), known as the “trace map” such that

\[
\gamma_0(u) = u|_{\partial \Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).
\]

So, the trace map extends the notion of boundary values to all Sobolev functions. We have

\[
\text{im} \gamma_0 = W^{1,p}(\partial \Omega) \left( \frac{1}{p} + \frac{1}{p'} = 1 \right) \text{ and } \ker \gamma_0 = W^{1,p}_0(\Omega).
\]

The trace map \( \gamma_0 \) is compact into \( L^q(\partial \Omega) \) for all \( q \in \left[ 1, \frac{(N-1)p}{N-1} \right) \) if \( N > p \) and into \( L^q(\partial \Omega) \) for all \( q \geq 1 \) if \( p \geq N \). In the sequel, for the sake of notational simplicity, we drop the use of the trace map \( \gamma_0(\cdot) \). All restrictions of Sobolev functions on \( \partial \Omega \) are understood in the sense of traces.

For every \( r \in (1, +\infty) \), let \( A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^* \) be defined by

\[
\langle A_r(u), h \rangle = \int_{\Omega} |Du|^{-2}(Du, Dh)_{\mathbb{R}^N} \, dz \text{ for all } u, h \in W^{1,r}(\Omega).
\]

The following proposition summarizes the main properties of this map (see Gasinski & Papageorgiou [3]).

**Proposition 1.** The map \( A_r(\cdot) \) is bounded (that is, it maps bounded sets to bounded sets) continuous, monotone (hence maximal monotone, too) and of type \((S)_+\), that is, if \( u_n \rightharpoonup u \) in \( W^{1,r}(\Omega) \) and \( \limsup_{n \to \infty} \langle A_r(u_n), u_n - u \rangle \), then \( u_n \to u \) in \( W^{1,r}(\Omega) \).

Evidently, the \((S)_+\)-property is useful in verifying the C-condition.

Now we introduce the conditions on the potential function \( \xi(\cdot) \) and on the boundary coefficient \( \beta(\cdot) \).

\begin{itemize}
  \item \( H(\xi) \): \( \xi \in L^\infty(\Omega) \) and \( \xi(z) \geq 0 \) for almost all \( z \in \Omega \).
  \item \( H(\beta) \): \( \beta \in C^{0,\alpha}(\partial \Omega) \) with \( 0 < \alpha < 1 \) and \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \).
  \item \( H_0 \): \( \xi \not\equiv 0 \) or \( \beta \not\equiv 0 \).
\end{itemize}

**Remark 1.** When \( \beta \equiv 0 \) we have the usual Neumann problem.

The next two propositions can be found in Papageorgiou & Rădulescu [15].

**Proposition 2.** If \( \xi \in L^\infty(\Omega) \), \( \xi(z) \geq 0 \) for almost all \( z \in \Omega \) and \( \xi \not\equiv 0 \), then \( c_0\|u\|^p \leq \|Du\|_p^p + \int_\Omega \xi(z)|u|^p \, dz \) for some \( c_0 > 0 \) and all \( u \in W^{1,p}(\Omega) \).

**Proposition 3.** If \( \beta \in L^\infty(\partial \Omega) \), \( \beta(z) \geq 0 \) for \( \sigma \)-almost all \( z \in \partial \Omega \) and \( \beta \not\equiv 0 \), then \( c_1\|u\|^p \leq \|Du\|_p^p + \int_{\partial \Omega} \beta(z)|u|^p \, d\sigma \) for some \( c_1 > 0 \) and all \( u \in W^{1,p}(\Omega) \).
In what follows, let \( \gamma_p : W^{1,p}(\Omega) \to \mathbb{R} \) be defined by
\[
\gamma_p(u) = \|Du\|_p^p + \int_{\Omega} \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).
\]

If hypotheses \( H(\xi), H(\beta), H_0 \) hold, then from Propositions 2 and 3 we can infer that
\[
c_2 \|u\|^p \leq \gamma_p(u) \quad \text{for some } c_2 > 0 \text{ and all } u \in W^{1,p}(\Omega).
\] (1)

As we have already mentioned in the Introduction, our approach also involves truncation and comparison techniques. So, the next strong comparison principle, a slight variation of Proposition 4 of Papageorgiou & Smyrlis [7], will be useful.

**Proposition 4.** If \( \hat{\xi} \in L^\infty(\Omega) \) with \( \hat{\xi}(z) \geq 0 \) for almost all \( z \in \Omega, h_1, h_2 \in L^\infty(\Omega), \)
\[
0 < c_3 \leq h_2(z) - h_1(z) \quad \text{for almost all } z \in \Omega,
\]
and the functions \( u_1, u_2 \in C^1(\overline{\Omega}) \setminus \{0\}, u_1 \leq u_2, u_1^- \gamma, u_2^- \gamma \in L^\infty(\Omega) \) satisfy
\[
-\Delta_p u_1 - \Delta_q u_1 + \hat{\xi}(z) u_1^{p-1} - u_1^- \gamma = h_1 \quad \text{for almost all } z \in \Omega,
\]
\[
-\Delta_p u_2 - \Delta_q u_2 + \hat{\xi}(z) u_2^{p-1} - u_2^- \gamma = h_2 \quad \text{for almost all } z \in \Omega,
\]
then \( u_2 - u_1 \in \text{int} \hat{\mathcal{C}}_+. \)

Consider a Carathéodory function \( f_0 : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfying
\[
|f_0(z,x)| \leq a_0(z)[1 + |x|^{r-1}] \quad \text{for almost all } z \in \Omega \text{ and all } x \in \mathbb{R},
\]
with \( a_0 \in L^\infty(\Omega) \) and \( 1 < r \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases} \) (the critical Sobolev exponent corresponding to \( p \)).

We set \( F_0(z,x) = \int_0^x f_0(z,s)ds \) and consider the \( C^1 \)-functional \( \varphi_0 : W^{1,p}(\Omega) \to \mathbb{R} \) defined by
\[
\varphi_0(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} F_0(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega) \text{ (recall that } q < p). \]

The next proposition can be found in Papageorgiou & Rădulescu [16] and essentially is an outgrowth of the nonlinear regularity theory of Lieberman [17].

**Proposition 5.** If \( u_0 \in W^{1,p}(\Omega) \) is a local \( C^1(\overline{\Omega}) \)-minimizer of \( \varphi_0 \), that is, there exists \( \rho_0 > 0 \) such that
\[
\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,
\]
then \( u_0 \in C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0,1) \) and \( u_0 \) is also a local \( W^{1,p}(\Omega) \)-minimizer of \( \varphi_0 \), that is, there exists \( \rho_1 > 0 \) such that
\[
\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } \|h\| \leq \rho_1.
\]

The next fact about ordered Banach spaces is useful in producing upper bounds for functions and can be found in Gasinski & Papageorgiou [18, p. 680] (Problem 4.180).

**Proposition 6.** If \( X \) is an ordered Banach space with positive (order) cone \( K \),
\[
\text{int } K \neq \emptyset \text{ and } e \in \text{int } K
\]
then for every \( u \in X \) we can find \( \lambda_u > 0 \) such that \( \lambda_u e - u \in K \).
Under hypotheses $H(\xi), H(\beta), H_0$, the differential operator $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with the Robin boundary condition, has a principal eigenvalue $\hat{\lambda}_1(p) > 0$ which is isolated, simple and admits the following variational characterization:

$$\hat{\lambda}_1(p) = \inf \left\{ \frac{\gamma_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right\}.$$  \hspace{1cm} (2)

The infimum is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. By $\hat{u}_1(p)$ we denote the positive, $L^p$-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1(p) > 0$. The nonlinear Hopf theorem (see, for example, Gasinski & Papageorgiou [3, p. 738]) implies that $\hat{u}_1(p) \in D_+.$

Let us fix some basic notation which we will use throughout this work. So, if $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and the for $u \in W^{1,p}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W^{1,p}(\Omega), \ u = u^+ - u^-, \ |u| = u^+ + u^-.$$  

If $\varphi \in C^1(W^{1,p}(\Omega), \mathbb{R})$, then by $K_\varphi$ we denote the critical set of $\varphi$, that is,

$$K_\varphi = \{ u \in W^{1,p}(\Omega) : \varphi'(u) = 0 \}.$$  

Also, if $u, y \in W^{1,p}(\Omega)$, with $u \leq y$, then we define

$$[u, y] = \{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \leq y(z) \text{ for almost all } z \in \Omega \},$$

$$[u] = \{ h \in W^{1,p}(\Omega) : u(z) \leq h(z) \text{ for almost all } z \in \Omega \},$$

$${\text{int}}_{C^1(\overline{\Omega})}[u, y] = \text{the interior in the } C^1(\overline{\Omega})\text{-norm of } [u, y] \cap C^1(\overline{\Omega}).$$

Now we introduce our hypotheses on the perturbation $f(z, x)$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $f(z, x) \leq a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$ and all $x \geq 0$ with $a \in L^\infty(\Omega), p < r < p^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x \to +\infty} \frac{F(z, x)}{x^p} = +\infty$ uniformly for almost all $z \in \Omega$;

(iii) there exists $\tau \in ((r-p) \max \left\{ \frac{N}{p}, 1 \right\}, p^*)$ such that

$$0 < \beta_0 \leq \liminf_{x \to +\infty} \frac{f(z, x)x - pF(z, x)}{x^\tau}$$

uniformly for almost all $z \in \Omega$;

(iv) for every $\vartheta > 0$, there exists $m_\vartheta > 0$ such that

$$m_\vartheta \leq f(z, x)$$

for almost all $z \in \Omega$ and all $x \geq \vartheta$;

(v) for every $\rho > 0$ and $\lambda > 0$, there exists $\tilde{\xi}_\rho^\lambda > 0$ such that for almost all $z \in \Omega$, the function $x \mapsto f(z, x) + \xi_\rho^\lambda x^{p-1}$ is nondecreasing on $[0, \rho]$.

**Remark 2.** Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis, without any loss of generality we may assume that

$$f(z, x) = 0 \text{ for almost all } z \in \Omega \text{ and all } x \leq 0.$$  \hspace{1cm} (3)

From hypotheses $H(f), (ii), (iii)$ it follows that

$$\lim_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty$$

uniformly for almost all $z \in \Omega$. 

Hence, for almost all $z \in \Omega$, the perturbation $f(z, \cdot)$ is $(p - 1)$-superlinear near $+\infty$. However, this superlinearity of $f(z, \cdot)$ is not expressed by using the well-known AR-condition. We recall that the AR-condition (unilateral version due to (3)) says that there exist $q > p$ and $M > 0$ such that
\begin{align*}
0 < q F(z, x) \leq f(z, x) x & \text{ for almost all } z \in \Omega \text{ and all } x \geq M, \\
0 < \inf_{\Omega} F(\cdot, M). & \quad (4a)
\end{align*}

Integrating (4a) and using (4b), we obtain the following weaker condition
\begin{align*}
c_4 x^q \leq F(z, x) & \text{ for almost all } z \in \Omega \text{ all } x \geq M, \text{ and some } c_4 > 0, \\
\Rightarrow c_4 x^{q-1} \leq f(z, x) & \text{ for almost all } z \in \Omega \text{ and all } x \geq M.
\end{align*}

So, the AR-condition dictates at least $(q - 1)$-polynomial growth for $f(z, \cdot)$. Here, we replace the AR-condition with hypothesis $H(f)(iii)$ which is less restrictive and permits superlinear nonlinearities with “slower” growth near $+\infty$. For example, the function
\[ f(x) = x^{p-1} \ln(1 + x) \text{ for all } x \geq 0. \]

(for the sake of simplicity we have dropped the $z$-dependence) satisfies hypotheses $H(f)$, but fails to satisfy the AR-condition.

We introduce the following sets:
\[ \mathcal{L} = \{ \lambda > 0 : \text{problem (P_{\lambda}) has a positive solution} \}, \]
\[ S_\lambda = \text{the set of positive solutions of (P_{\lambda})}. \]

Also we set
\[ \lambda^* = \sup \mathcal{L}. \]

3. Some auxiliary Robin problems

Let $\eta > 0$. First, we examine the following auxiliary Robin problem
\begin{align*}
\begin{cases}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \eta \text{ in } \Omega, \\
\frac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial \Omega, \ u > 0.
\end{cases}
\end{align*}

\[ (6) \]

**Proposition 7.** If hypotheses $H(\xi), H(\beta), H_0$ hold, then for every $\eta > 0$ problem (6) has a unique solution $\tilde{u}_\eta \in D_+$, the mapping $\eta \mapsto \tilde{u}_\eta$ is strictly increasing (that is, $\eta < \eta' \Rightarrow \tilde{u}_\eta - \tilde{u}_{\eta'} \in \text{int} \ C_+$) and
\[ \tilde{u}_\eta \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } \eta \to 0^+. \]

**Proof.** Consider the map $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by
\begin{align*}
\langle V(u), h \rangle &= \langle A_p(u), h \rangle + \langle A_q(u), h \rangle + \int_{\Omega} \xi(z)|u|^{p-2}uhdz + \int_{\partial \Omega} \beta(z)|u|^{p-2}uhd\sigma \\
&\quad \text{for all } u, h \in W^{1,p}(\Omega). \quad (7)
\end{align*}

Evidently, $V(\cdot)$ is continuous, strictly monotone (hence maximal monotone, too) and coercive (see (1)). Therefore $V(\cdot)$ is surjective (see Gasinski & Papageorgiou [3, Corollary 3.2.31, p. 319]). So, we can find $\tilde{u}_\eta \in W^{1,p}(\Omega), \tilde{u}_\eta \neq 0$ such that
\[ V(\tilde{u}_\eta) = \eta. \]

The strict monotonicity of $V(\cdot)$ implies that $\tilde{u}_\eta$ is unique. We have
\[ \langle V(\tilde{u}_\eta), h \rangle = \eta \int_{\Omega} hdz \text{ for all } h \in W^{1,p}(\Omega). \quad (8) \]
In (8) we choose \( h = -\tilde{u}_\eta^- \in W^{1,p}(\Omega) \). Then
\[
c_2 \|\tilde{u}_\eta^-\|^p \leq 0 \quad \text{(see (1))},
\]
\[
\Rightarrow \tilde{u}_\eta^- \geq 0, \quad \tilde{u}_\eta^- \neq 0.
\]
From (8) we have
\[
\begin{bmatrix}
-\Delta_p \tilde{u}_\eta(z) - \Delta_q \tilde{u}_\eta(z) + \xi(z)\tilde{u}_\eta(z)^{p-1} = \eta \\
\frac{\partial \tilde{u}_\eta}{\partial \nu_{pq}} + \beta(z)\tilde{u}_\eta^{p-1} = 0
\end{bmatrix}
\quad \text{on } \partial \Omega.
\]
From (9) and Proposition 7 of Papageorgiou & Rădulescu [16] we deduce that
\[
\tilde{u}_\eta \in L^\infty(\Omega).
\]
Then the nonlinear regularity theory of Lieberman [17] implies that
\[
\tilde{u}_\eta \in C_+ \setminus \{0\}.
\]
From (9) we have
\[
\Delta_p \tilde{u}_\eta(z) + \Delta_q \tilde{u}_\eta(z) \leq \|\xi\|_\infty \tilde{u}_\eta(z)^{p-1} \quad \text{for almost all } z \in \Omega,
\]
\[
\Rightarrow \tilde{u}_\eta \in D_+ \quad \text{(see Pucci & Serrin [19, pp. 111, 120])}.
\]
Suppose that \( 0 < \eta_1 < \eta_2 \) and let \( \tilde{u}_{\eta_1}, \tilde{u}_{\eta_2} \in D_+ \) be the corresponding solutions of problem (6). We have
\[
-\Delta_p \tilde{u}_{\eta_1} - \Delta_q \tilde{u}_{\eta_1} + \xi(z)\tilde{u}_1^{p-1} = \eta_1 < \eta_2 = -\Delta_p \tilde{u}_{\eta_2} - \Delta_q \tilde{u}_{\eta_2} + \xi(z)\tilde{u}_{\eta_2}
\quad \text{for almost all } z \in \Omega,
\]
\[
\Rightarrow \tilde{u}_{\eta_2} - \tilde{u}_{\eta_1} \in \mathcal{C}_+ \quad \text{(see Proposition 4)},
\]
\[
\Rightarrow \eta \mapsto \tilde{u}_\eta \text{ is strictly increasing from } (0, +\infty) \text{ into } C^1(\overline{\Omega}).
\]
Finally, let \( \eta_n \to 0^+ \) and let \( \tilde{u}_n = \tilde{u}_{\eta_n} \in D_+ \) be the corresponding solutions of (6). As before, invoking Proposition 7 of Papageorgiou & Rădulescu [16], we can find \( c_5 > 0 \) such that
\[
\|\tilde{u}_n\|_\infty \leq c_5 \quad \text{for all } n \in \mathbb{N}.
\]
Then from Lieberman [17] we infer that there exist \( \alpha \in (0, 1) \) and \( c_6 > 0 \) such that
\[
\tilde{u}_n \in C^{1,\alpha}(\overline{\Omega}), \quad \|\tilde{u}_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_6 \quad \text{for all } n \in \mathbb{N}.
\]
Exploiting the compact embedding of \( C^{1,\alpha}(\overline{\Omega}) \) into \( C^1(\overline{\Omega}) \), the monotonicity of the sequence \( \{\tilde{u}_n\}_{n \geq 1} \subseteq D_+ \) and the fact that for \( \eta = 0, u \equiv 0 \) is the only solution of (6) we obtain
\[
\tilde{u}_n \to 0 \text{ in } C^1(\overline{\Omega}).
\]
The proof is now complete. \( \square \)

Using Proposition 7, we see that we can find \( \eta_0 > 0 \) such that
\[
\eta \leq \tilde{u}_\eta(z)^{-\gamma} \quad \text{for all } z \in \overline{\Omega}, \quad 0 < \eta \leq \eta_0.
\]
We consider the following purely singular problem
\[
\begin{bmatrix}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma} \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu_{pq}} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial \Omega, \quad u > 0, \quad 0 < \gamma < 1.
\end{bmatrix}
\]
In the first place, by a solution of (11) we understand a weak solution, that is, a function $u \in W^{1,p}(\Omega)$ such that
\[
\langle A_p(u), h \rangle + \langle A_q(u), h \rangle + \int_{\Omega} \xi(z)|v|^{p-2}vhdz + \int_{\partial\Omega} \beta(z)|v|^{p-2}vhd\sigma = \int_{\Omega} u^{-\gamma}h \quad \text{for all } h \in W^{1,p}(\Omega).
\]

In fact, using the nonlinear regularity theory, we will be able to establish more regularity for the solution of (11), which in fact, is a strong solution (that is, the equation can be interpreted pointwise almost everywhere on $\Omega$).

**Proposition 8.** If hypotheses $H(\xi), H(\beta), H_0$ hold, then problem (11) admits a unique solution $v \in D_+$.

**Proof.** Let $\eta \in (0, \eta_0]$ (see (10)) and recall that $\tilde{u}_\eta \in D_+$. So $m_\eta = \min_{\Omega} \tilde{u}_\eta > 0$ and
\[
\eta \leq \tilde{u}_\eta^{-\gamma} \leq m_\eta^{-\gamma} \quad \text{(see (10))},
\]
\[
\Rightarrow \tilde{u}_\eta^{-\gamma} \in L^\infty(\Omega). \quad (12)
\]

We consider the following truncation of the reaction in problem (11):
\[
k(z, x) = \begin{cases} 
\tilde{u}_\eta(z)^{-\gamma} & \text{if } x \leq \tilde{u}_\eta(z) \\
 x^{-\gamma} & \text{if } \tilde{u}_\eta(z) < x.
\end{cases} 
\quad (13)
\]

This is a Carathéodory function. We set $K(z, x) = \int_{\xi}^{x} k(z, s)ds$ and consider the $C^1$-functional $\Psi : W^{1,p}(\Omega) \to \mathbb{R}$ defined by
\[
\Psi(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} ||Du||^q_q - \int_{\Omega} K(z, u)dz \quad \text{for all } u \in W^{1,p}(\Omega).
\]

From (12) and (13), we see that $\Psi(\cdot)$ is coercive. Also the Sobolev embedding theorem and the compactness of the trace map, imply that $\Psi(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $v \in W^{1,p}(\Omega)$ such that
\[
\Psi(v) = \inf\{ \Psi(u) : u \in W^{1,p}(\Omega) \},
\]
\[
\Rightarrow \Psi'(v) = 0,
\]
\[
\Rightarrow \langle A_p(v), h \rangle + \langle A_q(v), h \rangle + \int_{\Omega} \xi(z)|v|^{p-2}vhdz + \int_{\partial\Omega} \beta(z)|v|^{p-2}vhd\sigma = \int_{\Omega} k(z, v)hdz \quad \text{for all } h \in W^{1,p}(\Omega). \quad (14)
\]

In (14) we choose $(\tilde{u}_\eta - v)^+ \in W^{1,p}(\Omega)$. Then
\[
\langle A_p(v), (\tilde{u}_\eta - v)^+ \rangle + \langle A_q(v), (\tilde{u}_\eta - v)^+ \rangle + \int_{\Omega} \xi(z)|v|^{p-2}v(\tilde{u}_\eta - v)^+dz + \\
\int_{\partial\Omega} \beta(z)|v|^{p-2}v(\tilde{u}_\eta - v)^+d\sigma = \int_{\Omega} \tilde{u}_\eta^{-\gamma}(\tilde{u}_\eta - v)^+dz \quad \text{(see (13))}
\]
\[
\geq \int_{\Omega} \eta(\tilde{u}_\eta - v)^+dz \quad \text{(see (10) and recall that } 0 < \eta \leq \eta_0) \\
\Rightarrow \int_{\partial\Omega} \eta(\tilde{u}_\eta - v)^+d\sigma \quad \text{(see Proposition 7),}
\]
\[
\Rightarrow \tilde{u}_\eta \leq v. \quad (15)
\]
Then from (13), (14), (15) we obtain
\[
\begin{cases}
-\Delta_p v(z) - \Delta_q v(z) + \xi(z)v(z)^{p-1} = v(z)^{-\gamma} \text{ for almost all } z \in \Omega, \\
\frac{\partial v}{\partial n_{pq}} + \beta(z)v^{p-1} = 0 \text{ on } \partial \Omega
\end{cases}
\] (16)

(see Papageorgiou & Rădulescu [20]).

From (15) we have \(v^{-\gamma} \leq \tilde{u}_n^{-\gamma} \in L^\infty(\Omega)\) (see (12)). So, from (16) and (15) we have \(v \in L^\infty(\Omega)\). Then the nonlinear regularity theory of Lieberman [17] implies that \(v \in C_+\). Hence it follows from (15) that
\[
v \in D_+.
\]

Next, we show that this positive solution is unique. To this end, let \(\tilde{v} \in W^{1,p}(\Omega)\) be another positive solution of (11). Again we have \(\tilde{v} \in D_+\). Then
\[
\langle A_p(v), (\tilde{v} - v)^+ \rangle + \langle A_q(v), (\tilde{v} - v)^+ \rangle + \int_\Omega \xi(z)v^{p-1}(\tilde{v} - v)^+ dz + \int_{\partial \Omega} \beta(z)v^{p-1}(\tilde{v} - v)^+ d\sigma
\]
\[
= \int_\Omega v^{-\gamma}(\tilde{v} - v)^+ dz
\]
\[
\geq \int_\Omega \tilde{v}^{-\gamma}(\tilde{v} - v)^+ dz
\]
\[
= \langle A_p(\tilde{v}), (\tilde{v} - v)^+ \rangle + \langle A_q(\tilde{v}), (\tilde{v} - v)^+ \rangle + \int_\Omega \xi(z)\tilde{v}^{p-1}(\tilde{v} - v)^+ dz + \int_{\partial \Omega} \beta(z)\tilde{v}^{p-1}(\tilde{v} - v)^+ d\sigma
\]
\[
\Rightarrow \tilde{v} \leq v.
\]

Interchanging the roles of \(v\) and \(\tilde{v}\) in the above argument, we obtain
\[
v \leq \tilde{v},
\]
\[
\Rightarrow v = \tilde{v}.
\]

This proves the uniqueness of the positive solution of the purely singular problem (11). \(\square\)

Next, we consider the following nonlinear Robin problem
\[
\begin{cases}
-\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma} + 1 \text{ in } \Omega, \\
\frac{\partial u}{\partial n_{pq}} + \beta(z)u^{p-1} = 0 \text{ on } \partial \Omega, \ u > 0.
\end{cases}
\] (17)

**Proposition 9.** If hypotheses \(H(\xi), H(\beta), H_0\) hold, then problem (17) admits a unique solution \(\pi \in D_+\) and \(v \leq \pi\).

**Proof.** We know that \(v^{-\gamma} \in L^\infty(\Omega)\) (see (12) and (15)). Then the existence and uniqueness of the solution \(\pi \in W^{1,p}(\Omega)\setminus\{0\}, \pi \geq 0\) of (17) follow from the surjectivity and strict monotonicity of the map \(V(\cdot)\) (see the proof of Proposition 7). The nonlinear regularity theory and the nonlinear Hopf’s theorem imply that \(\pi \in D_+\).

Moreover, we have
\[
\langle A_p(\pi), (\pi - v)^+ \rangle + \langle A_q(\pi), (\pi - v)^+ \rangle + \int_\Omega \xi(z)\pi^{p-1}(\pi - v)^+ dz + \int_{\partial \Omega} \beta(z)\pi^{p-1}(\pi - v)^+ d\sigma
\]
Proof. Let \( v \in D_+ \) be the unique positive solution of the auxiliary problem \((11)\) (see Proposition 8) and \( \overline{v} \in D_+ \) the unique solution of \((17)\) (see Proposition 9). We know that \( v \leq \overline{v} \) (see Proposition 9). Since \( \overline{v} \in D_+ \), hypothesis \( H(f)\) implies that

\[
0 \leq f(z, \overline{v}(z)) \leq c_7 \text{ for some } c_7 > 0 \text{ and almost all } z \in \Omega.
\]

So, we can find \( \lambda_0 > 0 \) so small that

\[
0 \leq \lambda f(z, \overline{v}(z)) \leq 1 \text{ for almost all } z \in \Omega \text{ and all } 0 < \lambda \leq \lambda_0.
\] (18)

We consider the following truncation of the reaction in problem \((P_\lambda)\)

\[
\vartheta_\lambda(z, x) = \begin{cases} 
v(z)^{-\gamma} + \lambda f(z, v(z)) & \text{if } x < v(z) \\
x^{-\gamma} + \lambda f(z, x) & \text{if } v(z) \leq x \leq \overline{v}(z) \\
\overline{v}(z)^{-\gamma} + \lambda f(z, \overline{v}(z)) & \text{if } \overline{v}(z) < x.
\end{cases}
\] (19)

This is a Carathéodory function. We set \( \theta_\lambda(z, s) = \int_0^s \vartheta_\lambda(z, s) ds \) and consider the functional \( \mu_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) \((\lambda \in (0, \lambda_0])\) defined by

\[
\mu_\lambda(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|^q_q - \int_\Omega \theta_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).
\]

Since \( 0 \leq \overline{v}^{-\gamma} \leq v^{-\gamma} \in L^\infty(\Omega) \), we see that \( \mu_\lambda \in C^1(W^{1,p}(\Omega)) \). Also, it is clear from \((1)\) and \((19)\), that \( \mu_\lambda(\cdot) \) is coercive. In addition, it is sequentially weakly lower semicontinuous. So, we can find \( u_\lambda \in W^{1,p}(\Omega) \) such that

\[
\mu_\lambda(u_\lambda) = \inf \{ \mu_\lambda(u) : u \in W^{1,p}(\Omega) \},
\]

\[
\Rightarrow \mu'_\lambda(u_\lambda) = 0,
\]

\[
\Rightarrow \langle A_p(u_\lambda), h \rangle + \langle A_q(u_\lambda), h \rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda hdz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma
\]

\[
= \int_\Omega \vartheta_\lambda(z, u_\lambda) hdz \text{ for all } h \in W^{1,p}(\Omega).
\] (20)
In (20) first we choose \( h = (u_\lambda - \overline{u})^+ \in W^{1,p}(\Omega) \). Then
\[
\langle A_p(u_\lambda), (u_\lambda - \overline{u})^+ \rangle + \langle A_q(u_\lambda), (u_\lambda - \overline{u})^+ \rangle + \int_{\Omega} \xi(z)u_\lambda^{p+}(u_\lambda - \overline{u})^+dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(u_\lambda - \overline{u})d\sigma \\
= \int_{\Omega} [\overline{u}^{-\gamma} + \lambda f(z, \overline{u})](u_\lambda - \overline{u})^+dz \text{ (see (19))} \\
\leq \int_{\Omega} [\overline{u}^{-\gamma} + 1](u_\lambda - \overline{u})^+dz \text{ (see (18))} \\
\leq \int_{\Omega} [v^{-\gamma} + 1](u_\lambda - \overline{u})^+dz \text{ (since } v \leq \overline{u}) \\
= \langle A_p(\overline{u}), (u_\lambda - \overline{u})^+ \rangle + \langle A_q(\overline{u}), (u_\lambda - \overline{u})^+ \rangle + \int_{\Omega} \xi(z)\overline{u}^{p-1}(u_\lambda - \overline{u})^+dz \\
+ \int_{\partial\Omega} \beta(z)\overline{u}^{p-1}(u_\lambda - \overline{u})^+d\sigma \text{ (see Proposition 9),} \\
\Rightarrow u_\lambda \leq \overline{u}.
\]

Next, in (20) we choose \( h = (v - u_\lambda)^+ \in W^{1,p}(\Omega) \). Then
\[
\langle A_p(u_\lambda), (v - u_\lambda)^+ \rangle + \langle A_q(u_\lambda), (v - u_\lambda)^+ \rangle + \int_{\Omega} \xi(z)|u_\lambda|^{p-2}u_\lambda(v - u_\lambda)^+dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda(v - u_\lambda)^+d\sigma \\
= \int_{\Omega} [v^{-\gamma} + \lambda f(z, v)](v - u_\lambda)^+dz \text{(see (19))} \\
\geq \int_{\Omega} v^{-\gamma}(v - u_\lambda)^+dz \text{(since } f \geq 0) \\
= \langle A_p(v), (v - u_\lambda)^+ \rangle + \langle A_q(v), (v - u_\lambda)^+ \rangle + \int_{\Lambda} \xi(z)v^{p-1}(v - u_\lambda)^+dz \\
+ \int_{\partial\Omega} \beta(z)v^{p-1}(v - u_\lambda)^+d\sigma \text{ (see Proposition 8),} \\
\Rightarrow v \leq u_\lambda.
\]

So, we have proved that
\[
u_\lambda \in [v, \overline{u}]. \tag{21}
\]

From (19), (20), (21) it follows that
\[
\begin{cases}
-\Delta_p u_\lambda(z) - \Delta_q u_\lambda(z) + \xi(z)u_\lambda(z)^{p-1} = u_\lambda(z)^{-\gamma} + \lambda f(z, u_\lambda(z)) \\
\text{for almost all } z \in \Omega, \\
\frac{\partial u_\lambda}{\partial n_{pq}} + \beta(z)u_\lambda^{p-1} = 0 \text{ on } \partial\Omega, \text{ (see [20]).} 
\end{cases} \tag{22}
\]

By (22) and Proposition 7 of Papageorgiou & Rădulescu [16], we have that \( u_\lambda \in L^\infty(\Omega) \). So, the nonlinear regularity theory of Lieberman [17] implies that \( u_\lambda \in D_+ \text{ (see (21))}. \) Therefore we have proved that
\[
(0, \lambda_0) \subseteq \mathcal{L} \neq \emptyset \text{ and } S_\lambda \subseteq D_+.
\]

The proof is now complete. \( \square \)
Next, we establish a lower bound for the elements of $S_\lambda$.

**Proposition 11.** If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, $\lambda \in \mathcal{L}$ and $u \in S_\lambda$, then $v \leq u$.

**Proof.** From Proposition 10 we know that $u \in D_+$. Then Proposition 7 implies that for $\eta > 0$ small enough, we have $\tilde{u}_\eta \leq u$. So, we can define the following Carathéodory function

$$e(z, x) = \begin{cases} 
\tilde{u}_\eta(z)^{-\gamma} & \text{if } x < \tilde{u}_\eta(z) \\
x^{-\gamma} & \text{if } \tilde{u}_\eta(z) \leq x \leq u(z) \\
u(z)^{-\gamma} & \text{if } u(z) < x.
\end{cases} \quad (23)$$

We set $E(z, x) = \int_0^x e(z, s)ds$ and consider the functional $d : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$d(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \| Du \|^q_q - \int_\Omega E(z, u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

As before, we have $d \in C^1(W^{1,p}(\Omega))$. Also, $d(\cdot)$ is coercive (see (23)) and weakly lower semicontinuous. Hence, we can find $\hat{v} \in W^{1,p}(\Omega)$ such that

$$d(\hat{v}) = \inf \{ d(u) : u \in W^{1,p}(\Omega) \},$$

$$d'(\hat{v}) = 0,$$

$$\Rightarrow (A_p(\hat{v}), h) + (A_q(\hat{v}), h) + \int_\Omega \xi(z)|\hat{v}|^{p-2}\hat{v}hdz + \int_{\partial \Omega} \beta(z)|\hat{v}|^{p-2}\hat{v}d\sigma = \int_\Omega e(z, \hat{v})hdz \text{ for all } h \in W_{1,p}(\Omega). \quad (24)$$

In (24) first we choose $h = (\hat{v} - u)^+ \in W^{1,p}(\Omega)$. Exploiting the fact that $u \in S_\lambda$ and recalling that $f \geq 0$, we obtain $\hat{v} \leq u$. Next, in (24) we test with $h = (\tilde{u}_\eta - v)^+ \in W^{1,p}(\Omega)$. Using (23), (10) and Proposition 7, we obtain $\tilde{u}_\eta \leq \hat{v}$. Therefore

$$\hat{v} \in [\tilde{u}_\eta, u]. \quad (25)$$

From (23), (24), (25) and Proposition 8, we conclude that

$$\hat{v} = v,$$

$$\Rightarrow v \leq u \text{ for all } u \in S_\lambda.$$

The proof is now complete. □

Now we can deduce a structural property of $\mathcal{L}$.

**Proposition 12.** If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold, $\lambda \in \mathcal{L}$, $0 < \mu < \lambda$ and $u_\lambda \in S_\lambda \subseteq D_+$, then $\mu \in \mathcal{L}$ and we can find $u_\mu \in S_\mu \subseteq D_+$ such that $u_\lambda - u_\mu \in \text{ int } \hat{C}_+$.

**Proof.** From Proposition 11 we know that $v \leq u_\lambda$. Therefore we can define the following Carathéodory function

$$\hat{k}_\mu(z, x) = \begin{cases} 
\hat{v}(z)^{-\gamma} + \mu f(z, \hat{v}(z)) & \text{if } x < \hat{v}(z) \\
x^{-\gamma} + \mu f(z, x) & \text{if } \hat{v}(z) \leq x \leq u_\lambda(z) \\
u_\lambda(z)^{-\gamma} + \mu f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x.
\end{cases} \quad (26)$$

We set $\hat{K}_\mu(z, x) = \int_0^x \hat{k}_\mu(z, s)ds$ and consider the $C^1$-functional $\hat{\Psi}_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\Psi}_\mu(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \| Du \|^q_q - \int_\Omega \hat{K}_\mu(z, u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

12
Evidently, \( \hat{\mu}_\mu(\cdot) \) is coercive (see (26)) and sequentially weakly lower semicontinuous. So, we can find \( u_\mu \in W^{1,p}(\Omega) \) such that
\[
\hat{\mu}_\mu(u_\mu) = \inf \left\{ \hat{\mu}_\mu(u) : u \in W^{1,p}(\Omega) \right\},
\]
implies
\[
\Rightarrow \hat{\mu}_\mu(u_\mu) = 0,
\]
implies
\[
\Rightarrow \langle A_p(u_\mu), h \rangle + \langle A_q(u_\mu), h \rangle + \int_\Omega \xi(\xi(z)) |u_\mu|^{p-2} u_\mu h dz + \int_{\partial \Omega} \beta(z) |u_\mu|^{p-2} u_\mu h d\sigma
\]
\[
= \int_\Omega \hat{k}_u(z, u_\mu) h dz \quad \text{for all } h \in W^{1,p}(\Omega).
\]
(27)

In (27) first we choose \( h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega) \). Using (26), the fact that \( \mu < \lambda \) and that \( f \geq 0 \) and recalling that \( u_\lambda \in S_\lambda \), we conclude that \( u_\mu \leq u_\lambda \). Next, in (27) we choose \( h = (v - u_\mu)^+ \in W^{1,p}(\Omega) \). From (26), the fact that \( f \geq 0 \) and Proposition 8, we infer that \( v \leq u_\mu \). Therefore we have proved that
\[
u_\mu \in [v, u_\lambda].
\]
(28)

From (26), (27), (28) it follows that
\[
u_\mu \in S_\mu \subseteq D_+ \quad \text{(see Proposition 10)}.
\]

Let \( \rho = \|u_\lambda\|_\infty \) and let \( \hat{\xi}_\rho^\lambda > 0 \) be as postulated by hypothesis \( H(f)(v) \). We have
\[
-\Delta_p u_\lambda z - \Delta_q u_\mu z + \left[ \xi(z) + \hat{\xi}_\rho^\lambda \right] u_\mu z^{p-1} - u_\mu(z)^{-\gamma}
\]
\[
= \mu f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\mu(z)^{p-1}
\]
\[
= \lambda f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\mu(z)^{p-1} - (\lambda - \mu) f(z, u_\mu(z))
\]
\[
< \lambda f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} \quad \text{(recall that } \lambda > \mu)\]
\[
< \lambda f(z, u_\mu(z)) + \hat{\xi}_\rho^\lambda u_\lambda(z)^{p-1} \quad \text{(see (28) and hypothesis } H(f)(v)\}
\]
\[
= -\Delta_p u_\lambda z - \Delta_q u_\mu z + \left[ \xi(z) + \hat{\xi}_\rho^\lambda \right] u_\lambda z^{p-1} - u_\lambda(z)^{-\lambda} \quad \text{for almost all } z \in \Omega
\]
\[
(29)
\]
(29)

We know that
\[
0 \leq u_\mu^{-\gamma}, u_\mu^{-\gamma} \leq v^{-\gamma} \in L^\infty(\Omega).
\]

Also, from hypothesis \( H(f)(iv) \) and since \( u_\mu \in D_+ \), we have
\[
0 < c_8 \leq (\lambda - \mu) f(z, u_\mu(z)) \quad \text{for almost all } z \in \Omega.
\]

Invoking Proposition 4, from (29) we conclude that
\[
u_\lambda - u_\mu \in \text{int } \mathcal{C}_+.
\]

The proof is now complete. \( \square \)

**Proposition 13.** If hypotheses \( H(\xi), H(\beta), H_0, H(f) \) hold, then \( \lambda^* < +\infty \).

**Proof.** On account of hypotheses \( H(f)(i) \rightarrow (iv) \), we can find \( \lambda_0 > 0 \) so big that
\[
x^{-\gamma} + \lambda_0 f(z, x) \geq x^{p-1} \quad \text{for almost all } z \in \Omega \quad \text{and all } x \geq 0.
\]
(30)
Let $\lambda > \lambda_0$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq D_+$ (see Proposition 10). Then $m_\lambda = \min_{\Omega} u_\lambda > 0$. For $\delta \in (0, 1)$ we set $m_\lambda^\delta = m_\lambda + \delta$ and for $\rho = \|u_\lambda\|_\infty$ let $\xi_\rho^\delta > 0$ be as postulated by hypothesis $H(f)(v)$. We have

$$-\Delta_p m_\lambda^\delta - \Delta_q m_\lambda^\delta + [\xi(z) + \xi_\rho^\delta](m_\lambda^\delta)^{p-1} - (m_\lambda^\delta)^{-\gamma} = [\xi(z) + \xi_\rho^\delta]m_\lambda^{p-1} - m_\lambda^{-\gamma} + \chi(\delta) \text{ with } \chi(\delta) \to 0^+ \text{ as } \delta \to 0^+$$

$$\leq \lambda f(z, m_\lambda) + [\xi(z) + \xi_\rho^\delta]m_\lambda^{p-1} + \chi(\delta) \text{ (see (30))}$$

$$\leq \lambda f(z, u_\lambda) + [\xi(z) + \xi_\rho^\delta]u_\lambda^{p-1} + \chi(\delta)(\text{ see hypothesis } H(f)(v))$$

$$= \lambda f(z, u_\lambda) + [\xi(z) + \xi_\rho^\delta]u_\lambda^{p-1} - (\lambda - \lambda_0)f(z, u_\lambda) + \chi(\delta)$$

$$= \lambda f(z, u_\lambda) + [\xi(z) + \xi_\rho^\delta]u_\lambda^{p-1} \text{ for } \delta \in (0, 1) \text{ small}$$

(recall that $u_\lambda \in D_+$ and see $H(f)(iv)$).

$$= -\Delta_p u_\lambda - \Delta_q u_\lambda + [\xi(z) + \xi_\rho^\delta]u_\lambda^{p-1} - u_\lambda^{-\gamma}. \quad (31)$$

Since $(\lambda - \lambda_0)f(z, u_\lambda) - \chi(\delta) \geq c_9 > 0$ for almost all $z \in \Omega$ and for $\delta \in (0, 1)$ small (just recall that $u_\lambda \in D_+$ and use hypothesis $H(f)(iv)$), invoking Proposition 4, from (31) we infer that

$$u_\lambda - m_\lambda^\delta \in \text{int } \mathcal{C}_+ \text{ for all } \delta \in (0, 1) \text{ small enough.}$$

However, this contradicts the definition of $m_\lambda$. It follows that $\lambda \notin \mathcal{L}$ and so $\lambda^* \leq \lambda_0 < +\infty$. \(\square\)

Therefore we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*].$$

**Proposition 14.** If hypotheses $H(\xi), H(\beta), H_0, H(f)$ hold and $\lambda \in (0, \lambda^*)$, then problem $(P_\lambda)$ has at least two positive solutions

$$u_0, \, \hat{u} \in D_+, \, u_0 \neq \hat{u}.$$  

**Proof.** Let $0 < \mu < \lambda < \eta < \lambda^*$. According to Proposition 12, we can find $u_\eta \in S_\eta \subseteq D_+, \, u_0 \in S_\lambda \subseteq D_+$ and $u_\mu \in S_\mu \subseteq D_+$ such that

$$u_\eta - u_0 \in \text{int } \mathcal{C}_+ \text{ and } u_0 - u_\mu \in \text{int } \mathcal{C}_+,$$

$$\Rightarrow u_0 \in \text{int}_{C^1(\Omega)}[u_\mu, u_\eta]. \quad (32)$$

We introduce the following Carathéodory function

$$\tau_\lambda(z, x) = \begin{cases} 
\mu(z)^{-\gamma} + \lambda f(z, u_\mu(z)) & \text{if } x < \mu(z) \\
\mu(z)^{-\gamma} + \lambda f(z, x) & \text{if } u_\mu(z) \leq x \leq u_\eta(z) \\
u_\eta(z)^{-\gamma} + \lambda f(z, u_\eta(z)) & \text{if } u_\eta(z) < x.
\end{cases} \quad (33)$$

Set $\tilde{T}_\lambda(z, x) = \int_0^x \tau_\lambda(z, s) ds$ and consider the $C^1$-functional $\tilde{\Psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\Psi}_\lambda(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\lambda} \tilde{T}_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (33) and the nonlinear regularity theory, we can easily check that

$$K_{\tilde{\Psi}_\lambda} \subseteq [u_\mu, u_\eta] \cap D_+. \quad (34)$$

Also, consider the Carathéodory function

$$\tau_\lambda^*(z, x) = \begin{cases} 
\mu(z)^{-\gamma} + \lambda f(z, u_\mu(z)) & \text{if } x < \mu(z) \\
\mu(z)^{-\gamma} + \lambda f(z, x) & \text{if } u_\mu(z) < x.
\end{cases} \quad (35)$$
We set $T^\lambda_{\ast}(z,x) = \int_{0}^{x} \tau^\lambda_{\ast}(z,s)ds$ and consider the $C^1$-functional $\Psi^\lambda_{\ast} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi^\lambda_{\ast}(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|^q - \int_{\Omega} T^\lambda_{\ast}(z,u)dz$$

for all $u \in W^{1,p}(\Omega)$.

For this functional using (35), we show that

$$K_{\Psi^\lambda_{\ast}} \subseteq [u_\mu] \cap D_+. \quad (36)$$

From (33) and (35) we see that

$$\widetilde{\Psi}_{\lambda} \big|_{[u_\mu,u_\eta]} = \Psi^\lambda_{\ast} \big|_{[u_\mu,u_\eta]} \quad \text{and} \quad \tilde{\Psi}'_{\lambda} \big|_{[u_\mu,u_\eta]} = (\Psi^\lambda_{\ast})' \big|_{[u_\mu,u_\eta]}. \quad (37)$$

From (34), (36), (37), it follows that without any loss of generality, we may assume that

$$K_{\Psi^\lambda_{\ast}} \cap [u_\mu,u_\eta] = \{u_0\}. \quad (38)$$

Otherwise it is clear from (35) and (36) that we already have a second positive smooth solution for problem $(P_{\lambda})$ and so we are done.

Note that $\tilde{\Psi}_{\lambda}(\cdot)$ is coercive (see (33)). Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\tilde{\Psi}_{\lambda}(\tilde{u}_0) = \inf \{ \tilde{\Psi}_{\lambda}(u) : u \in W^{1,p}(\Omega) \},$$

$$\Rightarrow \tilde{u}_0 \in K_{\tilde{\Psi}_{\lambda}},$$

$$\Rightarrow \tilde{u}_0 \in K_{\Psi^\lambda_{\ast}} \cap [u_\mu,u_\eta] \quad \text{(see (34),(37))},$$

$$\Rightarrow \tilde{u}_0 = u_0 \in D_+ \quad \text{(see (38))},$$

$$\Rightarrow u_0 \text{ is a local } C^1(\Omega)-\text{minimizer of } \Psi^\lambda_{\ast} \quad \text{(see (32))},$$

$$\Rightarrow u_0 \text{ is a local } W^{1,p}(\Omega)-\text{minimizer of } \Psi^\lambda_{\ast} \quad \text{(see Proposition 5)}.$$}

We assume that $K_{\Psi^\lambda_{\ast}}$ is finite. Otherwise on account of (35) and (36) we see that we already have an infinity of positive smooth solutions for problem $(P_{\lambda})$ and so we are done. Then (39) implies that we can find $\rho \in (0,1)$ small such that

$$\Psi^\lambda_{\ast}(u_0) < \inf \{ \Psi^\lambda_{\ast}(u) : \|u - u_0\| = \rho \} = m^\ast_{\lambda}$$

(see Papageorgiou, Rădulescu & Repovš [21, Theorem 5.7.6, p. 367]).

On account of hypothesis $H(f)(ii)$ we have

$$\Psi^\lambda_{\ast}(tu_1(p)) \to -\infty \text{ as } t \to +\infty. \quad (40)$$

Claim 1. $\Psi^\lambda_{\ast}(\cdot)$ satisfies the C- condition.

Let $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\Psi^\lambda_{\ast}(u_n)| \leq c_{10} \text{ for some } c_{10} > 0 \text{ and all } n \in \mathbb{N},$$

$$(1 + \|u_n\|)(\Psi^\lambda_{\ast})'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^{\ast}. \quad (41)$$

From (43) we have

$$|\langle A_p(u_n), h \rangle + \langle A_q(u_n), h \rangle + \int_{\Omega} \xi(z)|u_n|^{p-2}u_nhdz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_nhd\sigma$$

$$- \int_{\Omega} \tau^\lambda_{\ast}(z,u_n)hdz| \leq \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \text{ for all } h \in W^{1,p}, \text{ with } \epsilon_n \to 0^+. \quad (44)$$
Choosing $h = -u_n^- \in W^{1,p}(\Omega)$, we obtain
\[
\gamma_p(u_n^-) + \|Du_n^-\|_q^p \leq c_{11}\|u_n^-\| \text{ for some } c_{11} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (35))}
\]
\[
\Rightarrow \{u_n^-\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (1)) and recall that } 1 < p. \tag{45}
\]

Next in (44) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then
\[
- \gamma_p(u_n^+) - \|Du_n^+\|_q^p + \int_\Omega \tau(z, u_n) u_n^+ dz \leq \epsilon_n \text{ for all } n \in \mathbb{N},
\]
\[
\Rightarrow - \gamma_p(u_n^+) - \|Du_n^+\|_q^p + \int\left\{u_{\mu}^{-\gamma} + \lambda f(z, u_{\mu})\right\} u_n^+ dz \leq \epsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (35))} \tag{46}
\]

On the other hand from (42) and (45), we have
\[
\gamma_p(u_n^+) + \frac{p}{q} \|Du_n^+\|_p^q - \int\left\{p[u_{\mu}^{-\gamma} + \lambda f(z, u_{\mu})]\right\} u_n^+ dz \\
- \int\left\{u_{\mu} < u_n\right\} \left[\frac{p}{1-\gamma} (u_{\mu}^{-\gamma} - u_n^{-\gamma}) + p(\lambda F(z, u_{\mu}) - \lambda F(z, u_n))\right] dz \leq \epsilon_n
\]
\[
\text{for all } n \in \mathbb{N} \text{ (see (35)).}
\]
\[
\Rightarrow \gamma_p(u_n^+) + \frac{p}{q} \|Du_n^+\|_p^q - \int\left\{u_{\mu} \leq u_n\right\} p[u_{\mu}^{-\gamma} + \lambda f(z, u_{\mu})] u_n^+ dz \\
- \int\left\{u_{\mu} < u_n\right\} \left[\frac{p}{1-\gamma} u_n^{-\gamma} + \lambda p F(z, u_n)\right] dz \leq c_{12} \text{ for some } c_{12} > 0 \text{ and all } n \in \mathbb{N}. \tag{47}
\]

We add (46) and (47). Since $p > q$, we obtain
\[
\lambda \int\left\{u_{\mu} < u_n\right\} [f(z, u_n) u_n^+ - pF(z, u_n)] dz \leq (p-1) \int\left\{u_{\mu} \leq u_n\right\} [u_{\mu}^{-\gamma} + \lambda f(z, u_{\mu})] u_n^+ dz \\
+ \left(\frac{p}{1-\gamma} - 1\right) \int\left\{u_{\mu} < u_n\right\} u_n^{-\gamma} dz
\]
\[
\Rightarrow \lambda \int_{\Omega} [f(z, u_n^+) u_n^+ - pF(z, u_n^+)] dz \leq c_{13} \left[\|u_n^+\|_1 + 1\right] \tag{48}
\]
for some $c_{13} > 0$, all $n \in \mathbb{N}$.

On account of hypotheses $H(f)(i), (iii)$ we can find $\hat{\beta}_1 \in (0, \hat{\beta}_0)$ and $c_{14} > 0$ such that
\[
\hat{\beta}_1 x^\tau - c_{14} \leq f(z, x) - pF(z, x) \text{ for almost all } z \in \Omega \text{ and all } x \geq 0. \tag{49}
\]

Using (49) in (48), we obtain
\[
\|u_n^+\|_{\tau}^\tau \leq c_{15} \left[\|u_n^+\|_{\tau} + 1\right] \text{ for some } c_{15} > 0 \text{ and all } n \in \mathbb{N},
\]
\[
\Rightarrow \{u_n^+\}_{n \geq 1} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{50}
\]

First assume $N \neq p$. From hypothesis $H(f)(iii)$ it is clear that we may assume without any loss of generality that $\tau < r < p^\ast$. Let $t \in (0, 1)$ be such that
\[
\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^\ast}. 
\]
Then from the interpolation inequality (see Papageorgiou & Winkert [22, Proposition 2.3.17, p. 116]), we have
\[ \|u_n^+\|_r \leq \|u_n^+\|_r^{1-t}\|u_n^+\|_{p^*}^t, \]
\[ \Rightarrow \|u_n^+\|_r \leq c_{16}\|u_n^+\|_{p^*} \text{ for some } c_{16} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (50)).} \] (51)

From hypothesis $H(f)(i)$ we have
\[ f(z,x) < c_{17}[1 + x^r] \text{ for all } z \in \Omega, \text{ all } x \geq 0 \text{ and some } c_{17} > 0. \] (52)

From (44) with $h = u_n^+ \in W^{1,p}(\Omega)$, we obtain
\[ \gamma_p(u_n^+) + \|Du_n^+\|_q^2 - \int_{\Omega} \tau^\alpha(z,u_n)u_n^+dz \leq \epsilon_n \text{ for all } n \in \mathbb{N}, \]
\[ \Rightarrow \gamma_p(u_n^+) + \|Du_n^+\|_q^2 \leq \int_{\Omega} [(u_n^+)^{1-\gamma} + f(z,u_n)u_n^+]dz + c_{18} \]
for some $c_{18} > 0$ and all $n \in \mathbb{N}$ (see (54))
\[ \leq c_{19}[1 + \|u_n^+\|_{p^*}^r] \text{ for some } c_{19} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (52))} \]
\[ \leq c_{20}[1 + \|u_n^+\|_{p^*}^{r^*}] \text{ for some } c_{20} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (51)).} \] (53)

The hypothesis on $\tau$ (see $H(f)(iii)$) implies that $tr < p$. So, from (53) we infer that
\[ \{u_n^+\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded}, \]
\[ \Rightarrow \{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (45)).} \] (54)

If $N = p$, then $p^* = +\infty$ and from the Sobolev embedding theorem, we know that $W^{1,p}(\Omega) \hookrightarrow L^*(\Omega)$ for all $1 \leq s < \infty$. Then in order for the previous argument to work, we replace $p^* = +\infty$ by $s > r > \tau$ and let $t \in (0,1)$ as before such that
\[ \frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{s}, \]
\[ \Rightarrow tr = \frac{s(r-\tau)}{s-\tau}. \]

Note that $\frac{s(r-\tau)}{s-\tau} \rightarrow r - \tau$ as $s \rightarrow +\infty$. But $r - \tau < p$ (see hypothesis H(f)(iii)). We choose $s > r$ big so that $tr < p$. Then again we have (54).

Because of (54) and by passing to a subsequence if necessary, we may assume that
\[ u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \] (55)

In (44) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (55). Then
\[ \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A_q(u_n), u_n - u \rangle] = 0, \]
\[ \Rightarrow \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A_q(u), u_n - u \rangle] \leq 0 \]
(since $A_q(\cdot)$ is monotone)
\[ \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle \leq 0, \]
\[ \Rightarrow u_n \rightarrow u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1).} \]

Therefore $\Psi^*_\lambda(\cdot)$ satisfies the C-condition. This proves the Claim.

Then (40), (41) and the Claim permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that
\[ \hat{u} \in K_{\Psi^*_\lambda} \leq [u_\mu] \cap D_+ (\text{see (36)}) \text{, } m^\lambda_\mu \leq \Psi^*_\lambda(\hat{u}) \text{ (see (40)).} \]

Therefore $\hat{u} \in D_+$ is a second positive solution of problem $(P_\lambda)$ ($\lambda \in (0,\lambda^*)$) distinct from $u_0 \in D_+$. \(\square\)
Proof. Let Proposition 15. If hypotheses \( H(\xi), H(\beta), H_0, H(f) \) hold, then \( \lambda^* \in \mathcal{L} \).

Next, we examine what can be said in the critical parameter \( \lambda^* \).

Proposition 15. If hypotheses \( H(\xi), H(\beta), H_0, H(f) \) hold, then \( \lambda^* \in \mathcal{L} \).

Proof. Let \( \{\lambda_n\}_{n \geq 1} \subseteq (0, \lambda^*) \) be such that \( \lambda_n < \lambda^* \). We can find \( u_n \in S_{\lambda_n} \subseteq D_+ \) for all \( n \in \mathbb{N} \).

We consider the following Carathéodory function

\[
\mu_n(z, x) = \begin{cases} 
\frac{v(z) - \lambda_n f(z, v(z))}{x^{\gamma}} + \lambda_n f(z, x) & \text{if } x \leq v(z) \\
\frac{1}{x} & \text{if } v(z) < x
\end{cases}
\]

(56)

We set \( M_n(z, x) = \int_0^x \mu_n(z, x) \, ds \) and consider the \( C^1 \)-functional \( j_n : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[
j_n(u) = \frac{1}{p} \varepsilon_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} M_n(z, u) \, dz \text{ for all } u \in W^{1,p}(\Omega).
\]

Also, we consider the following truncation of \( \mu_n(z, \cdot) \)

\[
\hat{\mu}_n(z, x) = \begin{cases} 
\mu_n(z, x) & \text{if } x \leq u_{n+1}(z) \\
\mu_n(z, u_{n+1}(z)) & \text{if } u_{n+1}(z) < x
\end{cases}
\]

(recall that \( v \leq u_{n+1} \) for all \( n \in \mathbb{N} \), see Proposition 11). This is a Carathéodory function. We set \( \hat{M}_n(z, x) = \int_0^x \hat{\mu}_n(z, s) \, ds \) and consider the \( C^1 \)-functional \( \hat{j}_n : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[
\hat{j}_n(u) = \frac{1}{p} \varepsilon_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} \hat{M}_n(z, u) \, dz \text{ for all } u \in W^{1,p}(\Omega).
\]

From \( (1), (56) \) and \( (57) \), it is clear that \( \hat{j}_n(\cdot) \) is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find \( \hat{u}_n \in W^{1,p}(\Omega) \) such that

\[
\hat{j}_n(\hat{u}_n) = \inf \left\{ \hat{j}_n(u) : u \in W^{1,p}(\Omega) \right\}.
\]

(58)

Then we have

\[
\hat{j}_n(\hat{u}_n) \leq \hat{j}_n(v)
\]

\[
\leq \frac{1}{p} \varepsilon_p(v) + \frac{1}{q} \|Dv\|_q^q - \frac{1}{1 - \gamma} \int_{\Omega} v^{1-\gamma} \, dz
\]

(see \( (56), (57) \) and recall that \( f \geq 0 \))

\[
\leq (A_p(v), v) + (A_q(v), v) - \int_{\Omega} v^{1-\gamma} \, dz = 0
\]

(see Proposition 8).

From \( (58) \) we have

\[
\hat{u}_n \in K_{\hat{j}_n} \subseteq [v, u_{n+1}] \cap D_+ \text{ for all } n \in \mathbb{N} \text{ (see } (57)\).
\]

(59)

Similarly, using \( (56) \) we obtain

\[
K_{j_n} \subseteq [v) \cap D_+.
\]

(60)

Note that

\[
J_n|_{[v, u_{n+1}]} = \hat{j}_n|_{[v, u_{n+1}]} \text{ and } J'_n|_{[v, u_{n+1}]} = \hat{j}'_n|_{[v, u_{n+1}]} \text{ (see } (56), (57)\).
\]

(61)

Then from \( (59), (60), (61) \), we have

\[
J_n(\hat{u}_n) \leq 0 \text{ for all } n \in \mathbb{N}
\]

(62)
\[(A_p(\hat{u}_n), h) + (A_q(\hat{u}_n), h) + \int_{\Omega} \xi(z)\hat{u}_n^{p-1}hdz + \int_{\partial\Omega} \beta(z)\hat{u}_n^{p-1}hd\sigma = \int_{\Omega} \mu_n(z, \hat{u}_n)hdz \tag{63}\]

for all \(h \in W^{1,p}(\Omega), \) all \(n \in \mathbb{N}\).

Using (62), (63) and reasoning as in the Claim in the proof of Proposition 14, we show that
\[
\{\hat{u}_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\]

So, we may assume that
\[
\hat{u}_n \overset{w}{\rightharpoonup} \hat{u}_* \text{ in } W^{1,p}(\Omega) \text{ and } \hat{u}_n \rightarrow \hat{u}_* \text{ in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \tag{64}\]

In (63) we choose \(h = \hat{u}_n - \hat{u}_* \in W^{1,p}(\Omega)\), pass to the limit as \(n \rightarrow \infty\) and use (64). Then as before (see the proof of Proposition 14), we obtain
\[
\hat{u}_n \rightarrow \hat{u}_* \text{ in } W^{1,p}(\Omega). \tag{65}\]

In (63) we pass to the limit as \(n \rightarrow \infty\) and use (65). Then
\[
(A_p(\hat{u}_*), h) + (A_q(\hat{u}_*), h) + \int_{\Omega} \xi(z)\hat{u}_*^{p-1}hdz + \int_{\partial\Omega} \beta(z)\hat{u}_*^{p-1}hd\sigma \\
= \int_{\Omega} \left[\hat{u}_*^{r-1} + \lambda^* f(z, \hat{u}_*)\right]hdz \text{ for all } h \in W^{1,p}(\Omega) \text{ (see (56), (61))}, \\
\Rightarrow \hat{u}_* \in S_{\lambda^*} \subseteq D_+ \text{ and so } \lambda^* \in \mathcal{L}.
\]

The proof is now complete. \(\square\)

From this proposition it follows that
\[
\mathcal{L} = (0, \lambda^*].
\]

The next bifurcation-type theorem summarizes our findings and provides a complete description of the dependence of the set of positive solutions of problem \((P_\lambda)\) on the parameter \(\lambda > 0\).

**Theorem 16.** If hypotheses \(H(\xi), H(\beta), H_0, H(f)\) hold, then there exists \(\lambda^* > 0\) such that
(a) for all \(\lambda \in (0, \lambda^*)\) problem \((P_\lambda)\) has at least two positive solutions
\[
\hat{u}_0, \; \hat{u} \in D_+, \; \hat{u}_0 \neq \hat{u};
\]
(b) for \(\lambda = \lambda^*\) problem \((P_\lambda)\) has at least one positive solution \(\hat{u}_* \in D_+;\)
(c) for all \(\lambda > \lambda^*\) problem \((P_\lambda)\) does not have any positive solutions.

**Acknowledgments**

This research was supported by the Slovenian Research Agency grants P1-0292, J1-8131, J1-7025, N1-0064, and N1-0083.

**References**


