Complex Variables and Elliptic Equations

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcov20

A Caffarelli–Kohn–Nirenberg-type inequality with variable exponent and applications to PDEs

Mihai Mihăilescu a b, Vicențiu Rădulescu a c & Denisa Stancu-Dumitru a

a Department of Mathematics, University of Craiova, 200585 Craiova, Romania
b Department of Mathematics, Central European University, 1051 Budapest, Hungary
c Institute of Mathematics ‘‘Simion Stoilow’’ of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Available online: 09 Sep 2011

To cite this article: Mihai Mihăilescu, Vicenţiu Rădulescu & Denisa Stancu-Dumitru (2011): A Caffarelli–Kohn–Nirenberg-type inequality with variable exponent and applications to PDEs, Complex Variables and Elliptic Equations, 56:7-9, 659-669

To link to this article: http://dx.doi.org/10.1080/17476933.2010.487212

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.tandfonline.com/page/terms-and-conditions

This article may be used for research, teaching and private study purposes. Any substantial or systematic reproduction, re-distribution, re-selling, loan, sub-licensing, systematic supply or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings,
demand or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.
A Caffarelli–Kohn–Nirenberg-type inequality with variable exponent and applications to PDEs†

Mihai Mihăilescuab*, Vicențiu Rădulescuc and Denisa Stancu-Dumitrua

aDepartment of Mathematics, University of Craiova, 200585 Craiova, Romania; 
bDepartment of Mathematics, Central European University, 1051 Budapest, Hungary; 
cInstitute of Mathematics ‘Simion Stoilow’ of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Communicated by A. Pankov

(Received 12 January 2010; final version received 16 April 2010)

Given \( \Omega \subset \mathbb{R}^N \) \( (N \geq 2) \) a bounded smooth domain we establish a Caffarelli–Kohn–Nirenberg type inequality on \( \Omega \) in the case when a variable exponent \( p(x) \), of class \( C^1 \), is involved. Our main result is proved under the assumption that there exists a smooth vector function \( \tilde{a} : \Omega \rightarrow \mathbb{R}^N \), satisfying \( \text{div} \tilde{a}(x) > 0 \) and \( \tilde{a}(x) \cdot \nabla p(x) = 0 \) for any \( x \in \Omega \). Particularly, we supplement a result by Fan et al. [X. Fan, Q. Zhang, and D. Zhao, Eigenvalues of \( p(x) \)-Laplacian Dirichlet problem, J. Math. Anal. Appl. 302 (2005), pp. 306–317] regarding the positivity of the first eigenvalue of the \( p(x) \)-Laplace operator. Moreover, we provide an application of our result to the study of degenerate PDEs involving variable exponent growth conditions.

Keywords: Caffarelli–Kohn–Nirenberg-type inequality; eigenvalue problem; degenerate elliptic equation; variable exponent; critical point

AMS Subject Classifications: 35D05; 35J60; 35J70; 58E05; 35P05

1. Introduction

In [1], Caffarelli et al. proved in the context of some more general inequalities, the following result: given \( p \in (1, N) \), for all \( u \in C^1_c(\Omega) \), there exists a positive constant \( C_{a,b} \) such that

\[
\left( \int_\Omega |x|^{-bq} |u|^q \, dx \right)^{p/q} \leq C_{a,b} \int_\Omega |x|^{-ap} |\nabla u|^p \, dx,
\]

where

\[-\infty < a < \frac{N-p}{p}, \quad a \leq b \leq a + 1, \quad q = \frac{Np}{N-p(1+a-b)},\]

*Corresponding author. Email: mmihailes@yahoo.com
†Dedicated to Professor V.V. Zhikov on the occasion of his 70th birthday.
and $\Omega \subseteq \mathbb{R}^N$ is an arbitrary open domain. Note that the Caffarelli–Kohn–Nirenberg inequality (1.1) reduces to the classical Sobolev inequality (if $a = b = 0$) and to the Hardy inequality (if $a = 0$ and $b = 1$). Inequality (1.1) proves to be an important tool in studying degenerate elliptic problems. It is also related to the understanding of some important phenomena, such as best constants, existence or nonexistence of extremal functions, symmetry properties of minimizers, compactness of minimizing sequences and concentration phenomena. We refer to [2–11] for relevant applications of the Caffarelli–Kohn–Nirenberg inequality.

In the years that followed this inequality was extensively studied (see, e.g. [2–4,11] and the references therein). An important consequence of the Caffarelli–Kohn–Nirenberg inequality is that it enabled the study of some degenerate elliptic equations which involve differential operators of the type

$$\text{div}(a(x)|\nabla u(x)|^p),$$

where $a(x)$ is a nonnegative function satisfying $\inf_{x} a(x) = 0$. Thus, the resulting operator is not uniformly elliptic and consequently some of the techniques that can be applied in solving equations involving uniformly elliptic operators fail in this new context. Degenerate differential operators involving a nonnegative weight that is allowed to have zeros at some points or even to be unbounded are used in the study of many physical phenomena related to equilibrium of anisotropic continuous media.

The goal of this article is to obtain inequalities of type (1.1) in the case when the constant $p$ is replaced by a function $p(x)$ of class $C^1$ and to use them in studying some degenerate elliptic equations involving variable exponent growth conditions. Our attempt will be considered in the context of bounded smooth domains $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$. Particularly, we supplement the result of [12, Theorem 3.3], regarding the positivity of the first eigenvalue of the $p(x)$-Laplace operator.

2. Variable exponent Lebesgue and Sobolev spaces

We recall some definitions and basic properties of the Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $p(x) : \overline{\Omega} \to (1, \infty)$ is a continuous function. For further information and proofs we refer to [13,14]. On the other hand, regarding applications of variable exponent Lebesgue and Sobolev spaces to PDEs we refer to [15] while for some physical motivations of such problems we remember the contributions of Rajagopal and Ruzicka [16], Ruzicka [17] and Zhikov [18].

For any continuous function $h : \overline{\Omega} \to (1, \infty)$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

Given $p(x) \in C(\overline{\Omega}, (1, \infty))$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u ; \text{u is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}.$$
$L^{p(x)}(\Omega)$ endowed with the Luxemburg norm, that is

$$|u|_{p(x)} = \inf \left\{ \mu > 0; \, \frac{1}{\mu} \int_{\Omega} |u(x)|^{p(x)} \, dx \leq 1 \right\},$$

is a reflexive Banach space.

If $p_1, p_2$ are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere in $\Omega$ then there exists the continuous embedding $L^{p_1(x)}(\Omega) \hookrightarrow L^{p_2(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder type inequality

$$\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^{-}} + \frac{1}{q^{-}} \right) |u|_{p(x)} |v|_{q(x)}$$

(2.1)

holds true.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)}: L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} \, dx.$$

If $(u_n), u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold true:

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$$

(2.2)

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$$

(2.3)

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \rho_{p(x)}(u_n - u) \to 0$$

(2.4)

$$|u_n|_{p(x)} \to \infty \Leftrightarrow \rho_{p(x)}(u_n) \to \infty.$$

(2.5)

Next, we define $W^{1,p(x)}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ under the norm

$$\|u\| = \|\nabla u\|_{p(x)}.$$

The space $(W^{1,p(x)}_0(\Omega), \| \cdot \|)$ is a separable and reflexive Banach space. We note that if $q \in C_+^{+}(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W^{1,p(x)}_0(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \frac{\lambda p(x)}{N-p(x)}$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$.

3. A Caffarelli–Kohn–Nirenberg-type inequality in bounded domains involving variable exponent growth conditions

Assume $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is an open, bounded and smooth set.

For each $x \in \Omega$, $x = (x_1, \ldots, x_N)$ and $i \in \{1, \ldots, N\}$ we denote

$$m_i = \inf_{x \in \Omega} x_i \quad M_i = \sup_{x \in \Omega} x_i.$$
For each \(i \in \{1, \ldots, N\}\) let \(a_i : [m_i, M_i] \rightarrow \mathbb{R}\) be functions of class \(C^1\). Particularly, the functions \(a_i\) are allowed to vanish.

Let \(\tilde{a} : \Omega \rightarrow \mathbb{R}^N\) be defined by

\[
\tilde{a}(x) = (a_1(x_1), \ldots, a_N(x_N)).
\]

We assume that there exists \(a_0 > 0\) a constant such that

\[
\text{div} \, \tilde{a}(x) \geq a_0 > 0 \quad \forall x \in \overline{\Omega}.
\]  

(3.1)

Next, we consider \(p : \overline{\Omega} \rightarrow (1, N)\) is a function of class \(C^1\) satisfying

\[
\tilde{a}(x) \cdot \nabla p(x) = 0 \quad \forall x \in \Omega.
\]  

(3.2)

We prove the following result:

**Theorem 1** Assume that \(\tilde{a}(x)\) and \(p(x)\) are defined as above and satisfy conditions (3.1) and (3.2). Then there exists a positive constant \(C\) such that

\[
\int_\Omega |u(x)|^{p(x)} \, dx \leq C \int_\Omega |\tilde{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)} \, dx \quad \forall u \in C^1_c(\Omega).
\]  

(3.3)

**Proof** The proof of Theorem 1 is inspired by the ideas in [19, Théorème 20.7].

Simple computations based on relation (3.2) show that for each \(u \in C^1_c(\Omega)\) the following equality holds true:

\[
\text{div}(|u(x)|^{p(x)} \tilde{a}(x)) = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( |u(x)|^{p(x)} a_i(x) \right)
\]

\[
= |u(x)|^{p(x)} \text{div} \, \tilde{a}(x)
\]

\[
+ \sum_{i=1}^N a_i(x) \left[ p(x)|u(x)|^{p(x)-2} u(x) \frac{\partial u}{\partial x_i} + |u(x)|^{p(x)} \log(|u(x)|) \frac{\partial p}{\partial x_i} \right]
\]

\[
= |u(x)|^{p(x)} \text{div} \, \tilde{a}(x) + p(x)|u(x)|^{p(x)-2} u(x) \nabla u(x) \cdot \tilde{a}(x)
\]

\[
+ |u(x)|^{p(x)} \log(|u(x)|) \nabla p(x) \cdot \tilde{a}(x)
\]

\[
= |u(x)|^{p(x)} \text{div} \, \tilde{a}(x) + p(x)|u(x)|^{p(x)-2} u(x) \nabla u(x) \cdot \tilde{a}(x).
\]

On the other hand, the flux-divergence theorem implies that for each \(u \in C^1_c(\Omega)\) we have

\[
\int_\Omega \text{div}(|u(x)|^{p(x)} \tilde{a}(x)) \, dx = \int_{\partial \Omega} |u(x)|^{p(x)} \tilde{a}(x) \cdot \vec{n} \, d\sigma(x) = 0.
\]

Using the above pieces of information we infer that for each \(u \in C^1_c(\Omega)\) it holds true

\[
\int_\Omega |u(x)|^{p(x)} \text{div} \, \tilde{a}(x) \, dx \leq p^+ \int_\Omega |u(x)|^{p(x)-1} |\nabla u(x)||\tilde{a}(x)| \, dx.
\]

Next, we recall that for each \(\epsilon > 0\), each \(x \in \Omega\) and each \(A, B \geq 0\) the following Young type inequality holds true (see, e.g. [20, the footnote on p. 56])

\[
A B \leq \epsilon A^{p(x)} + \frac{1}{\epsilon^{q(x)-1}} B^{q(x)}.
\]
We fix $\epsilon > 0$ such that

$$p^+ \epsilon < a_0,$$

where $a_0$ is given by relation (3.1).

The above facts and relation (3.1) yield

$$a_0 \int_{\Omega} |u(x)|^{p(x)} \, dx \leq p^+ \left[ \epsilon \int_{\Omega} |u(x)|^{p(x)} \, dx + \int_{\Omega} \left( \frac{1}{\epsilon} \right)^{p(x)-1} |\tilde{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)} \, dx \right],$$

for any $u \in C^1_0(\Omega)$, or

$$(a_0 - \epsilon p^+) \int_{\Omega} |u(x)|^{p(x)} \, dx \leq \left[ \left( \frac{1}{\epsilon} \right)^{p(x)-1} + \left( \frac{1}{\epsilon} \right)^{p(x)+1} \right] p^+ \int_{\Omega} |\tilde{a}(x)|^{p(x)} |\nabla u(x)|^{p(x)} \, dx,$$

for any $u \in C^1_0(\Omega)$. The conclusion of Theorem 1 is now clear.

**Remark 1** The result of Theorem 1 implies the fact that under the hypotheses (3.1) and (3.2) there exists a positive constant $D$ such that

$$\int_{\Omega} |u(x)|^{p(x)} \, dx \leq D \int_{\Omega} |\nabla u(x)|^{p(x)} \, dx \quad \forall u \in C^1_0(\Omega).$$

Thus, we deduce that in the hypothesis of Theorem 1 we have

$$\inf_{u \in W^{p(x)}_0(\Omega), [0]} \frac{\int_{\Omega} |\nabla u|^{p(x)} \, dx}{\int_{\Omega} |u|^{p(x)} \, dx} > 0. \quad (3.4)$$

The above relation asserts that in this case the first eigenvalue of the $p(x)$-Laplace operator (i.e. $\Delta_{p(x)} u := \text{div}(|\nabla u|^{p(x)-2} \nabla u)$) is positive. That fact is not obvious as Fan et al. pointed out in [12]. Actually, the infimum of the set of eigenvalues corresponding to the $p(x)$-Laplace operator can be 0 [12, Theorem 3.1]. On the other hand, a necessary and sufficient condition such that (3.4) holds true has not been obtained yet except the case when $N = 1$ (in that case, the infimum is positive if and only if $p(x)$ is a monotone function, see [12, Theorem 3.2]). However, the authors of [12] pointed out that in the case $N > 1$ a sufficient condition to have (3.4) is a vector $l \in \mathbb{R}^N \setminus \{0\}$ to exist such that, for any $x \in \Omega$, the function $f(t) = p(x + tl)$ is monotone, for $t \in I_x := \{x + sl \in \Omega \}$ [12, Theorem 3.3]. Assuming $p$ is of class $C^1$ the monotony of function $f$ reads as follows: either

$$\nabla p(x + tl) \cdot l \geq 0, \quad \text{for all } t \in I_x, \ x \in \Omega,$$

or

$$\nabla p(x + tl) \cdot l \leq 0, \quad \text{for all } t \in I_x, \ x \in \Omega.$$
Example 1 We point out an example of functions $\tilde{a}(x)$ and $p(x)$ satisfying conditions (3.1) and (3.2) in the case when $\tilde{a}(x)$ can vanish in some points of $\Omega$. Let $N \geq 3$ and $\Omega = B_1(0)$, the ball centred in the origin of radius $\frac{1}{\sqrt{N}}$. We define $\tilde{a}(x) : \Omega \to \mathbb{R}^N$ by

$$\tilde{a}(x) = (-x_1, x_2, x_3, \ldots, x_{N-1}, x_N),$$

(more exactly, function $\tilde{a}(x)$ is associated with a vector $x \in \Omega$ the vector obtained from $x$ by changing in the first position $x_1$ by $-x_1$ and keeping unchanged $x_i$ for $i \in \{2, \ldots, N\}$). Clearly, $\tilde{a}(x)$ is of class $C^1$, $\tilde{a}(0) = 0$ and we have

$$\text{div}(\tilde{a}(x)) = N - 2 \geq 1 \quad \forall x \in \Omega.$$

Thus, condition (3.1) is satisfied.

Next, we define $p : \overline{\Omega} \to (1, N)$ by

$$p(x) = x_1(x_2 + x_3 + \cdots + x_{N-1} + x_N) + 2 \quad \forall x \in \overline{\Omega}.$$ 

It is easy to check that $p$ is of class $C^1$ and some elementary computations show that

$$\nabla p(x) \cdot \tilde{a}(x) = (x_2 + \cdots + x_N) (-x_1) + x_1x_2 + \cdots + x_1x_N = 0 \quad \forall x \in \Omega.$$ 

It means that condition (3.2) is satisfied, too.

Example 2 We point out a second example, for $N = 2$. Taking $\Omega = B_{\frac{1}{\sqrt{3}}} (0)$, $\tilde{a}(x) = (-x_1, 2x_2)$ and $p(x) = x_1^2x_2 + \frac{3}{2}$ it is easy to check that relations (3.1) and (3.2) are fulfilled.

Remark 2 If $N$, $a$ and $p$ are as in Example 1 or Example 2 then the result of Theorem 1 reads as follows: there exists a positive constant $C > 0$ such that

$$\int_{\Omega} |u(x)|^{p(x)} \, dx \leq C \int_{\Omega} |u(x)|^{p(x)} |\nabla u(x)|^{p(x)} \, dx \quad \forall u \in C^1_c(\Omega).$$  \hspace{1cm} (3.5)

4. Applications in solving PDEs involving variable exponent growth conditions

In this section we assume that $N$, $\Omega$, $\tilde{a}(x)$ and $p(x)$ are as in Example 1 or Example 2. We denote by $\mathcal{D}_0^{1, p(x)}(\Omega)$ the closure of $C^1_c(\Omega)$ under the norm

$$\|u\| = \|x\| |\nabla u(x)|_{p(x)}.$$ 

Undoubtedly, $(\mathcal{D}_0^{1, p(x)}(\Omega), \| \cdot \|)$ is a reflexive Banach space.

4.1. A compact embedding result

We prove the following result:

**Theorem 2** Assume that $N$, $\Omega$, $\tilde{a}(x)$ and $p(x)$ are as in Example 1 or Example 2 and $p^- > \frac{2N}{2N-1}$. Then $\mathcal{D}_0^{1, p(x)}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for each $q \in (1, \frac{2Np^-}{2N+p^-})$.

**Proof** Let $\{u_n\}$ be a bounded sequence in $\mathcal{D}_0^{1, p(x)}(\Omega)$. There exists $\epsilon_0 \in (0, 1)$ such that we have $\overline{B}_{\epsilon_0}(0) \subset \Omega$. Let $\epsilon \in (0, \epsilon_0)$ be arbitrary but fixed. By Theorem 1 it follows that $\{u_n\}$ is a bounded sequence in $L^{p(x)}(\Omega)$. Consequently, $\{u_n\} \subset W^{1, p(x)}(\Omega \setminus \overline{B}_\epsilon(0))$.
is a bounded sequence. Since \( W^{1,p(x)}(\Omega \setminus \overline{B}(0)) \subset W^{1,p(x)}(\Omega \setminus \overline{B}(0)) \) we deduce that \( \{u_n\} \) is a bounded sequence in \( W^{1,p(x)}(\Omega \setminus \overline{B}(0)) \). The classical compact embedding theorem shows that there exists a convergent subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), in \( L^q(\Omega \setminus \overline{B}(0)) \). Thus, for any \( n \) and \( m \) large enough we have

\[
\int_{\Omega \setminus \overline{B}(0)} |u_n - u_m|^q \, dx < \epsilon. \tag{4.1}
\]

On the other hand, the Hölder inequality for variable exponent spaces implies

\[
\int_{B,(0)} |u_n - u_m|^q \, dx = \int_{B,(0)} |x|^{-\frac{q}{p}} |x|^q |u_n - u_m|^q \, dx \\
\leq D_1 \left( \sup_{x \in B(0)} |x|^{-\frac{q}{p}} \right)^{\frac{q}{p}} \left( \int_{B(0)} |x|^q |u_n - u_m|^q \, dx \right)^{\frac{1}{q}},
\]

where \( D_1 \) is a positive constant.

Furthermore, inequality (3.5) and relations (2.2) and (2.3) imply

\[
| |x|^q |u_n - u_m|^q |_{\frac{q}{p}} \leq \left( \int_{\Omega} |x|^{q \frac{(p(x))} {p(x) - q}} |u_n - u_m|^{p(x)} \, dx \right)^{\frac{1}{p(x)}} + \left( \int_{\Omega} |x|^q |u_n - u_m|^q \, dx \right)^{\frac{1}{q}}
\]

\[
\leq \left[ \left( \sup_{x \in \Omega} |x| + 1 \right)^{\frac{q}{p(x)}} + \left( \int_{\Omega} |x|^q |u_n - u_m|^q \, dx \right)^{\frac{1}{q}} \right]^{\frac{1}{p(x)}} \left( \rho_{p(x)}(u_n - u_m) \right)^{\frac{1}{p(x)}}
\]

\[
\leq D_2 \left[ \left( \int_{\Omega} |x|^q |\nabla (u_n - u_m)|^{p(x)} \, dx \right)^{\frac{1}{q}} + \left( \int_{\Omega} |x|^q |\nabla (u_n - u_m)|^{p(x)} \, dx \right)^{\frac{1}{q}} \right],
\]

where \( D_2 \) is a positive constant.

Combining the above pieces of information we find that there exists a positive constant \( M \) such that

\[
\int_{B,(0)} |u_n - u_m|^q \, dx \leq M \left( \left( \frac{p(x)} {q} \right) \right)^{\frac{1}{p(x)}} \left( \left( \frac{p(x)} {q} \right) \right)^{\frac{1}{p(x)}}.
\]

But using again relations (2.2) and (2.3) it is easy to see that

\[
| |x|^{-\frac{q}{p}} \chi_{B,(0)} |_{\frac{q}{p}} \leq \left( \frac{p(x)} {q} \right)^{\frac{1}{p(x)}} \left( \left( \frac{p(x)} {q} \right) \right)^{\frac{1}{p(x)}} + \left( \frac{p(x)} {q} \right)^{\frac{1}{p(x)}} \left( \left( \frac{p(x)} {q} \right) \right)^{\frac{1}{p(x)}}
\]

where \( (\frac{p(x)} {q})' = \frac{p(x)} {p(x) - q} \), and assuming \( \epsilon \in (0,1) \)

\[
\int_{B,(0)} |x|^\frac{q p(x) - q}{q p(x) - q} \, dx \leq \int_{B,(0)} |x|^\frac{q p(x) - q}{q p(x) - q} \, dx
\]

\[
= \int_{0}^{\infty} \omega_N r^{N-1} r^{\frac{q p(x) - q}{q p(x) - q}} \, dr
\]

\[
= \omega_N \frac{1}{\alpha},
\]

where \( \alpha = \frac{\omega_N}{2(2p - q)} > 0 \) and \( \omega_N \) is the area of the unit ball in \( \mathbb{R}^N \).
Consequently,
\[
\int_{B_r(0)} |u_n - u_m|^q \, dx \leq M_1 (\epsilon^{\alpha_1} + \epsilon^{\alpha_2}),
\]
with \(\alpha_1, \alpha_2 > 0\) and \(M_1 > 0\) a constant.

The above inequality and relation (4.1) show that for any \(n\) and \(m\) large enough we have
\[
\int_{\Omega} |u_n - u_m|^q \, dx \leq M_2 (\epsilon + \epsilon^{\alpha_1} + \epsilon^{\alpha_2}),
\]
where \(M_2\) is a positive constant. We infer that \(\{u_n\}\) is a Cauchy sequence in \(L^q(\Omega)\) and consequently \(D_0^{1,p(x)}(\Omega)\) is compactly embedded in \(L^q(\Omega)\). The proof of Theorem 2 is complete.

**Remark 3** The proof of Theorem 2 still holds true if we replace the space \(L^q(\Omega)\) by \(L^{q(x)}(\Omega)\), where \(q : \overline{\Omega} \to (1, \infty)\) is a continuous function satisfying \(1 < q^- \leq q^+ < \frac{2Np^-}{2N+p^-}\).

### 4.2. Existence of solutions for a singular PDE involving variable exponent growth conditions

Assume \(q(x)\) is a function satisfying the hypothesis given in Remark 3. We investigate the existence of solutions of the problem
\[
\begin{cases}
-\text{div}(\lvert x \rvert^{p(x)} \vert \nabla u(x) \rvert^{p(x)-2} \nabla u(x)) = \lambda \lvert u(x) \rvert^{q(x)-2} u(x) & \text{for } x \in \Omega, \\
u(x) = 0 & \text{for } x \in \partial \Omega,
\end{cases}
\]
for \(x \in \Omega\), (4.2)

where \(\lambda\) is a positive constant.

We say that \(u \in D_0^{1,p(x)}(\Omega)\) is a weak solution of problem (4.2) if
\[
\int_{\Omega} \lvert x \rvert^{p(x)} \lvert \nabla u \rvert^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\Omega} \lvert u \rvert^{q(x)-2} u v \, dx = 0 \quad \forall v \in D_0^{1,p(x)}(\Omega).
\]

We show the following existence result on problem (4.2):

**Theorem 3** For each \(\lambda > 0\) problem (4.2) has a nontrivial weak solution.

**Proof of Theorem 3** In order to prove Theorem 3 we define, for each \(\lambda > 0\), the energetic functional associated with problem (4.2), \(J_\lambda : D_0^{1,p(x)}(\Omega) \to \mathbb{R}\) by
\[
J_\lambda(u) = \int_{\Omega} \frac{|x|^{p(x)}}{p(x)} |\nabla u|^{p(x)} \, dx - \lambda \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx,
\]
for each \(u \in D_0^{1,p(x)}(\Omega)\). Standard arguments (see, e.g. [21]) show that \(J_\lambda \in C^1(D_0^{1,p(x)}(\Omega), \mathbb{R})\) and its derivative is given by
\[
\langle J_\lambda'(u), v \rangle = \int_{\Omega} |x|^{p(x)} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx - \lambda \int_{\Omega} |u|^{q(x)-2} u v \, dx,
\]
for all \(u, v \in D_0^{1,p(x)}(\Omega)\). We infer that \(u\) is a solution of problem (4.2) if and only if it is a critical point of \(J_\lambda\). Consequently, we concentrate our efforts on finding critical
The arguments to prove (a), (b) and (c) are detailed below.

(a) Similar arguments as in the proof of [21, Proposition 3.6 (ii)] can be used in order to obtain the fact that $J_\lambda$ is weakly lower semi-continuous.

(b) It is obvious that for any $u \in D_0^{1,p(x)}(\Omega)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+} \int_\Omega |x|^{p(x)} |\nabla u|^{p(x)} \, dx - \frac{\lambda}{q^-} \int_\Omega |u|^{q(x)} \, dx$$

$$\geq \frac{1}{p^+} \int_\Omega |x|^{p(x)} |\nabla u|^{p(x)} \, dx - \frac{\lambda}{q^-} (|u|_{q(x)}^{q^-} + |u|_{q(x)}^{q^+}).$$

If $\|u\| > 1$ the above inequality and Theorem 2 imply that there exists a positive constant $K$ such that

$$J_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^-} - \frac{K\lambda}{q^-} (\|u\|^{q^-} + \|u\|^{q^+}).$$

Taking into account that $1 < q^- < \frac{2Np^-}{2N+p^-} < p^-$ the above inequality shows that $\lim_{\|u\| \to \infty} J_\lambda(u) = \infty$, that is $J_\lambda$ is coercive.

On the other hand, it is clear that for any $u \in D_0^{1,p(x)}(\Omega)$ we have

$$J_\lambda(u) \geq \frac{1}{p^+} \min\{\|u\|^{p^-}, \|u\|^{p^+}\} - \frac{K\lambda}{q^-} (\|u\|^{q^-} + \|u\|^{q^+}),$$

and thus, we deduce that $J_\lambda$ is bounded from below.

(c) We fix $\phi \in C_0^1(\Omega), \phi \neq 0$. Then for each $t \in (0, 1)$ we have

$$J_\lambda(t\phi) = \int_\Omega |x|^{p(x)} |t^{p(x)} \phi|^{p(x)} \, dx - \lambda \int_\Omega q(x) |\phi|^{q(x)} \, dx$$

$$\leq t^{p^-} \int_\Omega |x|^{p(x)} |\nabla \phi|^{p(x)} \, dx - \lambda t^{q^+} \int_\Omega \frac{1}{q(x)} |\phi|^{q(x)} \, dx.$$}

Thus, there exist $L_1$ and $L_2$ two positive constants such that for each $t \in (0, 1)$ we have

$$J_\lambda(t\phi) \leq L_1 t^{p^-} - L_2 t^{q^+}.$$
a critical point of $J_\lambda$ and thus, a nontrivial weak solution of problem (4.2). The proof of Theorem 3 is complete.

Acknowledgements

The research of Mihai Mihăilescu was supported by the grant CNCSIS-UEFISCU, project number PN II-RU PD-117/2010 ‘Probleme nelineare modelate de operatori diferentiali neomogeni’. Vicenţiu Rădulescu has been supported by the grant CNCSIS PNII-79/2007 ‘Degenerate and Singular Nonlinear Processes’. Denisa Stancu-Dumitru was partially supported by the strategic grant POSDRU/88/1.5/S/49516 Project ID 49516 (2009), co-financed by the European Social Fund Investing in People, within the Sectorial Operational Programme Human Resources Development 2007–2013.

References