Small linear perturbations of fractional Choquard equations with critical exponent

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Abstract

We are concerned with the qualitative analysis of positive solutions to the fractional Choquard equation

\[
\begin{cases}
(\Delta)^s u + a(x)u = (I_\alpha * |u|^{2^*_s})|u|^{2^*_s-2}u, & x \in \mathbb{R}^N, \\
u \in D^{s,2}(\mathbb{R}^N), & x \in \mathbb{R}^N,
\end{cases}
\]

where \( I_\alpha(x) \) is the Riesz potential, \( s \in (0, 1) \), \( N > 2s \), \( 0 < \alpha < \min\{N, 4s\} \), and \( 2^*_s = \frac{2N-\alpha}{N-2s} \) is the fractional critical Hardy-Littlewood-Sobolev exponent. We first establish a nonlocal global compactness property in the framework of fractional Choquard equations. In the second part of this paper, we prove that the equation has at least one positive solution in the case of small perturbations of the potential that describes the linear term.

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1. Introduction

The Choquard equation

$$-\Delta u + u = \left( \frac{1}{|x|} * |u|^2 \right) u \quad \text{in} \quad \mathbb{R}^3,$$  \quad (1.1)

was first introduced in the pioneering work of Fröhlich [18] and Pekar [40] for the modeling of quantum polaron. This model corresponds to the study of free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarization that it creates on the medium (interaction of an electron with its own hole). In the approximation to Hartree-Fock theory of one component plasma, Choquard used equation (1.1) to describe an electron trapped in its own hole.

The Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac system. Such a model was proposed for boson stars and for the collapse of galaxy fluctuations of scalar field dark matter. We refer for details to Elgart and Schlein [17], Giulini and Großardt [22], Jones [24], and Schunck and Mielke [46]. Penrose [41,42] proposed equation (1.1) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon. As pointed out by Lieb [28], Choquard used equation (1.1) to study steady states of the one component plasma approximation in the Hartree-Fock theory. Classification of solutions of (1.1) was first studied by Ma and Zhao [32]. For a broad survey of Choquard equations we refer to Moroz and van Schaftingen [37] and references therein. We also refer to Cassani and Zhang [9], Mingqi, Rădulescu and Zhang [34] and Seok [47] as recent relevant contributions to the study of Choquard-type problems. For more results on classical Choquard equations, we refer to [1,8,11,12,21,29,30,32,36] and the references therein.

In recent years, the fractional Laplace operator has attracted much attention. The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arises in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, we refer to [4,20,26,27,43,44] and the monograph by Molica Bisci, Rădulescu and Servadei [35].

We note that the study of the existence, concentration and multiplicity of positive solutions for the fractional Choquard equations is not much in the literature. In [15,16], d’Avenia, Siciliano and Squassina studied the regularity, existence, nonexistence, symmetry as well as decays properties of weak solutions for the following fractional subcritical Choquard equation

$$(-\Delta)^s u + w u = (|x|^{-\alpha} * |u|^p)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$ \quad (1.2)

where \(w > 0\), a constant, \(s \in (0, 1)\), \(N > 2s\), \(0 < \alpha < \min\{N, 4s\}\), \(\frac{2N-\alpha}{N} < p < 2^{\ast}_{\alpha,s}\), and \(2^{\ast}_{\alpha,s} = \frac{2N-\alpha}{N-2s}\) is the fractional Hardy-Littlewood-Sobolev critical exponent, \((-\Delta)^s\) denotes the fractional Laplacian operator. Later, in [48], the authors studied the existence of groundstates for the fractional Choquard equation

$$(-\Delta)^s u + u = (|x|^{-\alpha} * F(u))f(u), \quad x \in \mathbb{R}^N,$$ \quad (1.3)

where the nonlinearity is subcritical and satisfies the general Berestycki-Lions type conditions.
Let us consider the fractional critical Choquard equation
\[
\begin{cases}
(-\Delta)^s u + a(x)u = \left( \int_{\Omega} \frac{|u|^{2^*_s}}{|x-y|^{\alpha}} \, dx \right) |u|^{2^*_s-2} u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]
(1.4)
where \(\Omega \subseteq \mathbb{R}^N\), \(a(x)\) is a potential function. When \(\Omega = \mathbb{R}^N\) and \(a(x) \equiv 0\), we have known that problem (1.4) has the positive solution
\[
\frac{C_{N,\alpha,s}}{(1 + |x|^2)^{\frac{N-2s}{2}}},
\]
where \(C_{N,\alpha,s}\) is the normalization constant depending only on \(N, \alpha\) and \(s\), and all positive solutions can be obtained by this one by translations and scale changes, see [25, 38]. However, if \(\Omega \neq \mathbb{R}^N\) or \(a(x) \neq 0\), the situation is not so simple. For example, if \(\Omega\) is a bounded \(C^{1,1}\) domain and \(a(x) = \lambda\), constant, as a consequence of the Pohozaev identity [38, 45], it follows
\[
-s\lambda \int_{\Omega} u^2 \, dx = \frac{\Gamma(s+1)^2}{2} \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot v) \, d\sigma,
\]
where \(\Gamma(\cdot)\) is the Gamma function. If \(\Omega\) is a bounded starshaped domain, i.e., \((x \cdot v) > 0\) on \(\partial\Omega\), then (1.4) has no nontrivial solution when \(\lambda \geq 0\), and has a nontrivial solution while \(\lambda < -\bar{\lambda}\) for some \(\bar{\lambda} > 0\), see [38] for more details. In the case \(\Omega = \mathbb{R}^N\) and \(a(x) = \lambda \neq 0\), constant, a generalized version of the Pohozaev identity [15, 36] associated to (1.4) reads as
\[
\frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx + \frac{N\lambda}{2} \int_{\mathbb{R}^N} |u|^2 \, dx = \frac{2N-\alpha}{22^*_s} \int_{\mathbb{R}^{2N}} \frac{|u(x)u(y)|^{2^*_s}}{|x-y|^{\alpha}} \, dx \, dy,
\]
which implies that
\[
s\lambda \int_{\mathbb{R}^N} |u|^2 \, dx = 0
\]
so there is no solution with \(\lambda \neq 0\). If \(\Omega = \mathbb{R}^N\) and \(a(x) \neq \lambda, \lambda \neq 0\), how to get nontrivial solutions for \((P)\) becomes more complicated.

After a bibliography review we have found only a paper related to \((P)\) that is due to Ma and Zhang [33]. In that paper Ma and Zhang considered the case \(a(x) = \lambda V(x) - \beta\), and established the existence and multiplicity of some solutions by variational methods if \(\beta > 0\) is appropriately small and, \(\gamma > 0\) large, and \(V(x) \in C(\mathbb{R}^N, \mathbb{R})\) is a nonnegative function such that \(\text{int}V^{-1}(0)\) is a nonempty bounded set with smooth boundary. We notice that the energy of solution obtained in [33] (also in [25, 38]) is less than the threshold value
\[
c_{h,l} := \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}. \]

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At this point a question arises naturally: If $\Omega = \mathbb{R}^N$ and $a(x) \neq \lambda, \lambda \neq 0$, a constant, can we get a nontrivial solution of (1.4) with energy higher than $c_{h,l}$?

In this paper, we stress our attention to this case, and give an affirmative answer. We consider the following fractional Choquard equation involving pure critical nonlinearity

$$
\begin{cases}
(-\Delta)^s u + a(x)u = (I_{\alpha} * |u|^{2_{\alpha,s}^*})|u|^{2_{\alpha,s}^* - 2}u, & x \in \mathbb{R}^N, \\
u \in D^{s,2}(\mathbb{R}^N), & u(x) > 0, x \in \mathbb{R}^N,
\end{cases}
$$

(P)

where $I_{\alpha} : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$
I_{\alpha}(x) = \frac{A_{\alpha}}{|x|^\alpha}, \quad A_{\alpha} = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{N-\alpha}{2}\right)\pi^{N/2}} 2^{N-\alpha},
$$

and $s \in (0, 1), N > 2s, 0 < \alpha < \min\{N, 4s\}$ and $2_{\alpha,s}^* = \frac{2N-\alpha}{N-2s}$. The fractional Laplacian $(-\Delta)^s u$, of a smooth function $u : \mathbb{R}^N \to \mathbb{R}$, is defined by

$$
(-\Delta)^s u(x) := C_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy,
$$

where P.V. is a commonly used abbreviation for “in the principal value sense”, and $C_{N,s}$ denotes the normalization constant. The work space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$
D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2s} (\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy < +\infty \right\},
$$

endowed with the norm

$$
\|u\|^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy,
$$

where $2s^* = 2N/(N - 2s)$ is the fractional critical Sobolev constant. According to Propositions 3.4 and 3.6 of [39], we have that,

$$
\|u\|^2 = \|(\Delta)^\frac{s}{2} u\|^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy,
$$

by omitting the normalization $C_{N,s}$. For an elementary introduction to the fractional Laplacian and fractional Sobolev spaces we refer the interested reader to [35], and to [5,6,23,35,52,53] and references therein for more results on fractional critical elliptic problems.

Before enunciating our theorem, we make the following assumptions on the function $a : \mathbb{R}^N \to \mathbb{R}^+$:

$(a_1)$ The function $a$ is positive on a set of positive measure.
\((a_2)\) \(a \in L^q(\mathbb{R}^N)\) for all \(q \in [p_1, p_2]\), where \(1 < p_1 < \frac{2N-\alpha}{4s-\alpha} < p_2 < \frac{N}{4s-N}\) if \(2s < N < 4s\).

\((a_3)\) The \(L^\frac{N}{4s-\alpha}(\mathbb{R}^N)\)-norm of \(a\) satisfies the following inequality

\[
|a|_{L^\frac{N}{4s-\alpha}(\mathbb{R}^N)} < \left(2^{\frac{4s-\alpha}{N-\alpha}} - 1\right) S \frac{(2s-N)((N-\alpha)(1-\alpha)+2s)+(2N-\alpha)2s}{2(N-\alpha+2s)},
\]

where

\[
S = \inf_{u \in D^{s,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2s}_{\alpha,s} dx\right)^{\frac{2}{2s}}}
\]  \hspace{1cm} (1.6)

is the fractional best Sobolev exponent.

We say that \(u : \mathbb{R}^N \to \mathbb{R}\) is a weak solution of problem (\(P\)), if \(u \in D^{s,2}(\mathbb{R}^N)\) and for all \(\varphi \in D^{s,2}(\mathbb{R}^N)\) we have

\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi dx + \int_{\mathbb{R}^N} a(x) u \varphi dx = \int_{\mathbb{R}^N} (I_\alpha \ast |u|^{2_{\alpha,s}}) |u|^{2_{\alpha,s} - 2} u \varphi dx.
\]

In order to state the main result, we consider the \(C^1\) class functional \(I : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}\) associated to problem (\(P\)) given by

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} a(x) u^2 dx - \frac{1}{2^{2_{\alpha,s}}} \int_{\mathbb{R}^N} (I_\alpha \ast |u|^{2_{\alpha,s}}) |u|^{2_{\alpha,s} - 2} dx,
\]

and for any \(v \in D^{s,2}(\mathbb{R}^N)\),

\[
I'(u)v = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v dx + \int_{\mathbb{R}^N} a(x) uv dx - \int_{\mathbb{R}^N} (I_\alpha \ast |u|^{2_{\alpha,s}}) |u|^{2_{\alpha,s} - 2} uv dx.
\]

Using the above notations we are able to state our main result.

**Theorem 1.1.** Assume that \((a_1) - (a_3)\) are satisfied, then problem (\(P\)) has a positive solution \(u_0 \in D^{s,2}(\mathbb{R}^N)\) such that

\[
\frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{\frac{2N-\alpha}{N-\alpha}} < I(u_0) < \frac{2}{4N - 2\alpha} S_{h,l}^{\frac{2N-\alpha}{N-\alpha}}.
\]  \hspace{1cm} (1.7)

The proof of Theorem 1.1 is inspired from the seminal paper due to Benci and Cerami [7], where the authors investigated the existence of positive solution for the following classical Schrödinger equation with critical exponent

\[
-\Delta u + a(x)u = |u|^{2_{\alpha,s} - 2} u, \quad x \in \mathbb{R}^N.
\]  \hspace{1cm} (1.8)
The authors used the properties of the solutions of the limit equation $-\Delta u = |u|^2^*-2 u$ in combination with the variational methods and arguments of degree theory, to obtain positive solution to (1.8). However, we remark that in the proof of Theorem 1.1, the ideas of Benci and Cerami [7] cannot be immediately applicable to our situation due to the double nonlocal framework caused by the fractional Laplacian operator $(-\Delta)^s$ and the convolution term $I_\alpha * |u|^{2^*/\alpha}$. Some refined estimates for this problem are very necessary and delicate. For instance, see Lemmas 3.1, 4.11 and Theorem 3.2 below. After the publication of [7], some authors studied related problem to (1.8), see [2,3,10,13] and references therein.

We note that, when $s = 1$, the classical Laplacian case, Alves and Figueiredo [3] considered problem $(P)$ and established a nonlocal version of global compactness lemma by applying a Cherrier type inequality (see Lemma 2.2 [3]). However, in the nonlocal case, $0 < s < 1$, the fractional type Cherrier inequality is not clear and not available in the literature, and so we can not make use of such kind of inequality to treat problem $(P)$. Simultaneously, the nonlocality caused by the nonlocal operator $(-\Delta)^s$, makes the arguments explored in [3] unable to be carried out straightforwardly in our case because some estimates become more subtle to be established.

Another novelty is that, different from [25,33,38], where the energy of the solution is less than the level $c_{h,t}$, we show the existence of high energy solution of $(P)$ satisfying (1.7). To this aim, we shall establish the version to $\mathbb{R}^N$ of Struwe’s global compactness result [50] for problem $(P)$ to study the behavior of Palais-Smale sequences, which may be useful also in other different context and has never appeared in the literature, to the best of our knowledge.

The paper is organized as follows. In Section 2 we give some preliminary results involving the limit problem. In Section 3, we establish a splitting theorem and show some compactness results involving the energy functional associated to $(P)$. In Section 4, we prove some technical lemmas and the proof of Theorem 1.1 is completed in Section 5. In the Appendix, we give a preliminary lemma which is useful in proving the convergence of integral with nonlocal term.

**Notation.** Throughout this paper, $|\cdot|_q$ denotes for the norm in $L^q(\mathbb{R}^N)$; $B_r(x)$ denotes the ball in $\mathbb{R}^N$ centered at $x$ with radius $r$; The letters $C, C_i, i = 1, 2, \cdots$, denote various positive constants whose exact values are irrelevant, and $u^\pm = \max\{\pm u, 0\}$.

**2. Limit problem**

In this section we give some preliminary results involving the limit problem that will be useful in our approach. To begin with, we recall that the key point to apply variational method for problem $(P)$ is the following Hardy-Littlewood-Sobolev inequality.

**Proposition 2.1.** ([29]) Let $t, r > 1$ and $0 < \alpha < n$ with $1/t + \alpha/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \alpha, r)$ independent of $f, h$ such that

$$\iint_{\mathbb{R}^{2n}} \frac{f(x)h(y)}{|x - y|^\alpha} dx dy \leq C(t, n, \alpha, r)|f|_t|h|_r. \quad (2.1)$$

If $t = r = \frac{2N}{2N - \alpha}$, then

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^\frac{\alpha}{2} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2} + \frac{\alpha}{2})} \left\{ \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(N)} \right\}^{-1 + \frac{\alpha}{2}}.$$
In this case there is equality in (2.1) if and only if $f \equiv (\text{constant})h$ and
\[ h(x) = A(\gamma^2 + |x - a|^2)^{-\frac{2N-\alpha}{2}} \]
for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

In general, let $f = h = |u|^q$, then by Hardy-Littlewood-Sobolev inequality we have,
\[
\iint_{\mathbb{R}^{2N}} \frac{|u(x)u(y)|^q}{|x - y|^\alpha} \, dx \, dy
\]
is well defined if $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying
\[
\frac{2}{t} + \frac{\alpha}{N} = 2.
\]
Therefore, for $u \in D^{s,2}(\mathbb{R}^N)$, by Sobolev embedding theorems, we must have
\[
\frac{2N - \alpha}{N} \leq q \leq \frac{2N - \alpha}{N - 2s}.
\]
From this, for $u \in D^{s,2}(\mathbb{R}^N)$, we get
\[
\left( \int_{\mathbb{R}^N} (I_\alpha * |u|^{2s}) |u(x)|^{2s} \, dx \right)^{\frac{1}{2s}} \leq A_{\alpha} C(N, \alpha) \frac{1}{2s} |u|_{2s}^{\frac{1}{2s}} \equiv D_{N,\alpha}^{\frac{1}{2s}} |u|_{2s}^{\frac{1}{2s}}.
\] (2.2)

Let $S_{h,l}$ be the best constant
\[
S_{h,l} = \inf_{u \in D^{s,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx}{\left( \int_{\mathbb{R}^N} (I_\alpha * |u|^{2s}) |u|^{2s} \, dx \right)^{\frac{1}{2s}}},
\] (2.3)
and $S$ be the best Sobolev constant
\[
S = \inf_{u \in D^{s,2}(\mathbb{R}^N), u \neq 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |u|^{2s} \, dx \right)^{\frac{2}{2s}}},
\] (2.4)

where $2^*_s = 2N/(N - 2s)$ is the fractional Sobolev critical exponent. It is well-known that $S_{h,l}$ and $S$ are both achieved if and only if $u$ is of the form
\[
C \left( \frac{\epsilon}{\epsilon^2 + |x - x_0|^2} \right)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N
\]
for some $x_0 \in \mathbb{R}^N$, $C > 0$ and $\epsilon > 0$ (see Theorem 2.15 of [16]). Moreover,
In particular, let

\[ U(x) = \frac{\beta_1}{(1 + |x|^2)^{(N-2s)/2}}, \]

where \( \beta_1 = \left( \frac{S^{N/(2s)} \Gamma(N)}{\pi^{N/2} \Gamma(N/2)} \right)^{\frac{N-2s}{2N}}. \)

It follows from [14,52] that

\[ S|U|^2 = \|U\|^2, \quad (2.6) \]

and moreover, we have

\[ \|U\|^2 = |U|^2 = \int_{\mathbb{R}^N} \frac{\beta_1^{2s}}{(1 + |x|^2)^N} dx = \beta_1^{2N} \frac{\pi^{N/2} \Gamma(N/2)}{\Gamma(N)} = S^N. \]

Put

\[ \widetilde{U}(x) = S^{(N-a)/(2N-a+2s)} D_{N,a}^{2s-N} U(x), \]

then \( \widetilde{U}(x) \) is the unique minimizer for \( S_{h,l} \) and satisfies

\[ (-\Delta)^s u = (I_\alpha \ast |u|^{2\alpha,s}) |u|^{2\alpha,s-2} u, \quad x \in \mathbb{R}^N, \quad (2.7) \]

and

\[ \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{U}|^2 dx = \int_{\mathbb{R}^N} (I_\alpha \ast |\widetilde{U}|^{2\alpha,s}) |\widetilde{U}|^{2\alpha,s} dx = S_{h,l}^{N-\alpha}. \quad (2.8) \]

Moreover, by the invariance of the scaling, we see that the functions

\[ \widetilde{U}_{\epsilon,b}(x) = \epsilon^{\frac{2s-N}{2}} \widetilde{U} \left( \frac{x-b}{\epsilon} \right) = S^{(N-a)/(2N-a+2s)} D_{N,a}^{2s-N} \epsilon^{\frac{2s-N}{2}} U \left( \frac{x-b}{\epsilon} \right) \]

\[ = S^{(N-a)/(2N-a+2s)} D_{N,a}^{2s-N} \frac{\beta_1^N}{(\epsilon^2 + |x-b|^2)^{N-2s}} \]

\[ := S^{(N-a)/(2N-a+2s)} D_{N,a}^{2s-N} U_{\epsilon,b}(x) \]

\[ = \frac{b_{\alpha,s} \epsilon^{\frac{N-2s}{2}}}{(\epsilon^2 + |x-b|^2)^{\frac{N-2s}{2}}}, \quad \forall \, \epsilon > 0, \quad \forall \, b \in \mathbb{R}^N, \]

solve equation (2.7) and satisfy (2.8), where
\[ b_{\alpha,s} = S^{\frac{(N-\alpha)(2s-N)}{4(N-\alpha+2s)}} D_{N,\alpha}^{\frac{2s-N}{2N-\alpha+2s}} \beta_1. \]  

Let us introduce the limit problem associated to problem \((P)\),

\[
\begin{align*}
(-\Delta)_s^\alpha u &= (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha-2} u, \quad x \in \mathbb{R}^N, \\
& \in D^{s,2}(\mathbb{R}^N), \quad u(x) > 0, \quad x \in \mathbb{R}^N, 
\end{align*}
\]

whose the functional associated \(I_\infty: D^{s,2}(\mathbb{R}^N) \to \mathbb{R}\) is given by

\[ I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)_s^\alpha u|^2 \, dx - \frac{1}{2^{2^*_s}} \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha} \, dx. \]

**Lemma 2.2.** Let \((u_n)\) be a \((PS)_c\) sequence for \(I_\infty\). Then

(i) The sequence \((u_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\).

(ii) If \(u_n \rightharpoonup u\) in \(D^{s,2}(\mathbb{R}^N)\), then \(I_\infty'(u) = 0\).

(iii) If \(c \in (-\infty, \frac{N+2s-\alpha}{4N-2\alpha} S_{h,1}^{(2N-\alpha)/N+2s-\alpha})\), then \(I_\infty\) satisfies the \((PS)_c\) condition, that is, up to subsequence, \(u_n \rightharpoonup u\) in \(D^{s,2}(\mathbb{R}^N)\).

**Proof.** By hypothesis \(I_\infty(u_n) \to c\) and \(I_\infty'(u_n) \to 0\), then there exists \(K > 0\) such that

\[ K + \|u_n\| \geq I_\infty(u_n) - \frac{1}{2^{2^*_s}} I_\infty'(u_n)u_n = \frac{N+2s-\alpha}{4N-2\alpha} \|u_n\|^2, \quad \forall n \in \mathbb{N}, \]

proving (i). Since \(u_n \rightharpoonup u\) in \(D^{s,2}(\mathbb{R}^N)\), up to a subsequence, we get

\[ u_n \to L^q_{\text{loc}}(\mathbb{R}^N), \quad q \in (2, 2^*_s) \]

and

\[ u_n(x) \to u \text{ a.e. in } \mathbb{R}^N. \]

By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear map from \(L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)\) to \(L^{\frac{2N}{\alpha}}(\mathbb{R}^N)\). Thus

\[ I_\alpha * |u_n|^{2^*_\alpha} \to I_\alpha * |u|^{2^*_\alpha} \text{ in } L^{\frac{2N}{\alpha}}(\mathbb{R}^N), \]

and for each \(\phi \in C_0^\infty(\mathbb{R}^N)\),

\[ |u_n|^{2^*_\alpha-2} u_n \phi \to |u|^{2^*_\alpha-2} u \phi \text{ in } L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \]

The above limit implies that

\[ \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_\alpha})|u_n|^{2^*_\alpha-2} u_n \phi \, dx \to \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_\alpha})|u|^{2^*_\alpha-2} u \phi \, dx. \]
and so, \( I'_\infty(u)\phi = 0 \) for all \( \phi \in C^\infty_0(\mathbb{R}^N) \). Now (ii) follows from the density of \( C^\infty_0(\mathbb{R}^N) \) in \( D^{s,2}(\mathbb{R}^N) \).

In order to prove (iii), we consider \( v_n = u_n - u \) and employing Brezis-Lieb lemma and Lemma 2.2 in [19] we get

\[
o_n(1) = I'_\infty(u_n)u_n = \|u_n\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |u_n|^{2^*_s})|u_n|^{2^*_s} \, dx \\
= \|v_n\|^2 + \|u\|^2 - \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_s})|v_n|^{2^*_s} \, dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^{2^*_s})|u|^{2^*_s} \, dx + o_n(1) \tag{2.11}
\]

Hence, up to a subsequence, we conclude

\[
0 \leq l = \lim_{n \to \infty} \|v_n\|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_s})|v_n|^{2^*_s} \, dx.
\]

Suppose by contradiction that, \( l > 0 \). From the inequality

\[
S_{h,l} \left[ \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_s})|v_n|^{2^*_s} \, dx \right] \frac{1}{2^*_s} \leq \|v_n\|^2,
\]

we obtain

\[
l \geq [S_{h,l}]^{\frac{1}{2^*_s}} \Rightarrow l \geq S_{h,l}^{\frac{2N-\alpha}{N-2\alpha}}. \tag{2.12}
\]

In view of

\[
I_\infty(u) = \left( \frac{1}{2} - \frac{1}{22*_{s}} \right) \|u\|^2 = \frac{N + 2s - \alpha}{4N - 2\alpha} \|u\|^2 \geq 0,
\]

and

\[
c = \frac{N + 2s - \alpha}{4N - 2\alpha} \|v_n\|^2 + I_\infty(u) + o_n(1), \tag{2.13}
\]

we derive that

\[
c = \frac{N + 2s - \alpha}{4N - 2\alpha} \|v_n\|^2 + I_\infty(u) + o_n(1) \geq \frac{N + 2s - \alpha}{4N - 2\alpha} \|v_n\|^2 + o_n(1)
\]

\[
= \frac{N + 2s - \alpha}{4N - 2\alpha} l \geq \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{\frac{2N-\alpha}{N-2\alpha}}.
\]
which is a contradiction. Hence \( l = 0 \) and 
\[
\| v_n \|^2 = \| u_n - u \|^2 \to 0.
\]

The proof is now complete. \( \Box \)

**Lemma 2.3.** If \( u \in D^{s,2}(\mathbb{R}^N) \) is a nodal solution of \((P)_\infty\), then 
\[
I_\infty(u) \geq \frac{2^{4s-a} N + 2s - \alpha}{4N - 2\alpha} s h, l \frac{2N-a}{2s-a}.
\]

**Proof.** Arguing as in the proof of Proposition 3.2 [21], for all \( t^+, t^- > 0 \), we have 
\[
I_\infty(t^+u^+) + I_\infty(t^-u^-) \leq I_\infty(u) \frac{2^{4s-a} N + 2s - \alpha}{4N - 2\alpha} s h, l \frac{2N-a}{2s-a}.
\]

Fixing \( t^+, t^- > 0 \) such that \( I_\infty'(t^\pm u^\pm)(t^\pm u^\pm) = 0 \), it follows that 
\[
I_\infty(t^+ u^+) \geq \frac{N + 2s - \alpha}{4N - 2\alpha} s h, l \frac{2N-a}{2s-a}.
\]

The last two inequalities imply that 
\[
I_\infty(u) \geq \frac{2^{4s-a} N + 2s - \alpha}{4N - 2\alpha} s h, l \frac{2N-a}{2s-a},
\]
completing the proof. \( \Box \)

### 3. A nonlocal global compactness lemma

We start this section by establishing the following technical lemma for \( I_\infty \) which will be useful to prove our compactness theorem.

**Lemma 3.1.** (Nonlocal global compactness property) Let \((u_n)\) be a \((PS)_c\) sequence for the functional \( I_\infty \) with \( u_n \rightharpoonup 0 \) and \( u_n \not\to 0 \). Then there exist a sequence \((R_n) \subset \mathbb{R}^+\) and a nontrivial solution \( v_0 \in D^{s,2}(\mathbb{R}^N) \) of \((P_\infty)\) such that, up to a subsequence of \((u_n)\), we have that 
\[
w_n(x) = u_n(x) - R^{(N-2s)/2} v_0(R_n(x - x_n)) + o_n(1),
\]
is a \((PS)_{c-I_\infty(v_0)}\) sequence for \( I_\infty \).

**Proof.** Let \((u_n) \subset D^{s,2}(\mathbb{R}^N)\) be a \((PS)_c\) sequence for \( I_\infty \), i.e., 
\[
I_\infty(u_n) \to c \quad \text{and} \quad I_\infty'(u_n) \to 0.
\] (3.1)

From Lemma 2.2-(i) we know that \((u_n)\) is bounded in \( D^{s,2}(\mathbb{R}^N) \). Since \( u_n \rightharpoonup 0 \) and \( u_n \not\to 0 \), it follows from Lemma 2.2-(iii) that
\[ c \geq \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{\frac{2N - \alpha}{2s + \alpha}}. \]

Notice that
\[ c + o_n(1) = I_\infty(u_n) - \frac{1}{2^{2s}} I'\infty(u_n)u_n = \frac{N + 2s - \alpha}{4N - 2\alpha} \|u_n\|^2, \]

which implies that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx \geq S_{h,l}^{\frac{2N - \alpha}{2s + \alpha}}. \quad (3.2) \]

Let \( L \) be a number such that \( B_2(0) \) is covered by \( L \) balls of radius 1, \((R_n) \subset \mathbb{R}^+\), \((x_n) \subset \mathbb{R}^N\) such that
\[ \sup_{y \in \mathbb{R}^N} \int_{B_{R_n}^{-1}(y)} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{B_{R_n}^{-1}(x_n)} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \frac{S_{h,l}^{\frac{2N - \alpha}{2s + \alpha}}}{2L} \]

and
\[ v_n(x) = R_n^{(2s-N)/2} u_n \left( \frac{x}{R_n} + x_n \right). \]

Using a change of variable, we can show that
\[ \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx = \frac{S_{h,l}^{\frac{2N - \alpha}{2s + \alpha}}}{2L} = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx. \]

Now for each \( \Phi \in D^{s,2}(\mathbb{R}^N) \), we define the function
\[ \tilde{\Phi}_n(x) := \frac{R_n^{(N-2s)/2}}{2s} \Phi(R_n(x - x_n)) \]

which satisfies
\[ \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \tilde{\Phi}_n dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Phi dx, \quad (3.3) \]

and
\[ \int_{\mathbb{R}^N} (I_{\alpha} \ast |u_n|^{2_{\alpha,s}}) |u_n|^{2_{\alpha,s}} - 2 u_n \tilde{\Phi}_n dx = \int_{\mathbb{R}^N} (I_{\alpha} \ast |v_n|^{2_{\alpha,s}}) |v_n|^{2_{\alpha,s}} - 2 v_n \Phi dx. \quad (3.4) \]

These limits yield that
\[ I_\infty(v_n) \to c \quad \text{and} \quad I'_\infty(v_n) \to 0. \]

From Lemma 2.2, there exists \( v_0 \in D^{s,2}(\mathbb{R}^N) \) such that, up to a subsequence, \( v_n \rightharpoonup v_0 \) in \( D^{s,2}(\mathbb{R}^N) \) and \( I'_\infty(v_0) = 0 \).

As a consequence of Lions’s lemma [31], we have

\[
\int_{\mathbb{R}^N} |v_n|^{2^*_s} \phi dx \to \int_{\mathbb{R}^N} |v_0|^{2^*_s} \phi dx + \sum_{j \in J} \phi(x_j) v_j, \quad \forall \phi \in C_0^{\infty}(\mathbb{R}^N) \tag{3.5}
\]

and

\[
|(-\Delta)^{\frac{s}{2}} v_n|^2 \to \mu \geq |(-\Delta)^{\frac{s}{2}} v_0|^2 + \sum_{j \in J} \phi(x_j) \mu_j, \quad \forall \phi \in C_0^{\infty}(\mathbb{R}^N),
\]

for some \( \{x_j\} \subset \mathbb{R}^N \) and for some \( \{v_j\}_{j \in J}, \{\mu_j\}_{j \in J} \subset \mathbb{R}^+ \) with \( S v_j^{2^*_s} \leq \mu_j \), where \( J \) is at most a countable set. In fact, \( J \) is finite. To see this, consider \( \phi \in C_0^{\infty}(\mathbb{R}^N) \) such that \( 0 \leq \phi \leq 1 \), for all \( x \in \mathbb{R}^N \), \( \phi(x) = 0 \) for all \( x \in B_2(0) \) and \( \phi(x) = 1 \) for all \( x \in B_1(0) \). Now fix \( x_j \in \mathbb{R}^N \), \( j \in J \) and define \( \psi_\rho(x) = \phi(x - x_j/\rho) \), for each \( \rho > 0 \). Then \( 0 \leq \psi_\rho(x) \leq 1 \), for all \( x \in \mathbb{R}^N \), \( \psi_\rho(x) = 0 \) for all \( x \in B_2^{x_j}(x_j) \) and \( \psi_\rho(x) = 1 \) for all \( x \in B_1(x_j) \). We have that \( (v_n \psi_\rho) \) is bounded in \( D^{s,2}(\mathbb{R}^N) \) and \( I'_\infty(v_n) v_n \psi_\rho = o_n(1) \). Moreover, we have

\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} (v_n \psi_\rho) dx = \int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_s})|v_n|^{2^*_s} \psi_\rho dx + o_n(1). \tag{3.6}
\]

Notice that

\[
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} (v_n \psi_\rho) dx = \int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))^2 \psi_\rho(y)}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(\psi_\rho(x) - \psi_\rho(y)) v_n(x)}{|x - y|^{N+2s}} dx dy. \tag{3.7}
\]

It is easy to verify that

\[
\int_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))^2 \psi_\rho(y)}{|x - y|^{N+2s}} dx dy = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^{2^*_s} \psi_\rho(y) dy \to \int_{\mathbb{R}^N} \psi_\rho(y) d\mu \quad \text{as} \quad n \to +\infty
\]

and

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\[
\int_{\mathbb{R}^N} \psi_{\rho}(y) d\mu \to \mu(\{x_j\}) = \mu_j \quad \text{as} \quad \rho \to 0. \tag{3.8}
\]

Moreover, using Hölder’s inequality we obtain that

\[
\begin{aligned}
&\left\| \frac{v_n(x) - v_n(y)}{|x-y|^{N+2s}} \right\|_{L^2} \left\| \frac{\psi_{\rho}(x) - \psi_{\rho}(y)}{|x-y|^{N+2s}} \right\|_{L^2} \int_{\mathbb{R}^{2N}} |v_n(x)|^2 \, dx \, dy \\
\leq & C \left( \int_{\mathbb{R}^{2N}} |\psi_{\rho}(x) - \psi_{\rho}(y)|^2 |v_n(x)|^2 \, dx \, dy \right)^{\frac{1}{2}} \tag{3.9}
\end{aligned}
\]

By Lemma 3.6 in [53], we derive

\[
\lim_{\rho \to 0} \lim_{n \to +\infty} \int_{\mathbb{R}^{2N}} |\psi_{\rho}(x) - \psi_{\rho}(y)|^2 |v_n(x)|^2 \, dx \, dy = 0. \tag{3.10}
\]

On the other hand, using Proposition 2.1, we have, as \( n \to +\infty \),

\[
\int_{\mathbb{R}^N} \left( I_{\alpha} \ast |v_n|^{2^*_s}\right) |v_n|^{2^*_s} \psi_{\rho} \, dx \leq C \int_{\mathbb{R}^N} |v_n|^{2^*_s} \left( \int_{\mathbb{R}^N} |v_n|^{2^*_s} \psi_{\rho} \, dx \right)^{\frac{2N-\alpha}{2s}} \right)^{\frac{2^*_s}{2s}}
\]

and

\[
\int_{\mathbb{R}^N} \psi_{\rho}(y) d\nu \to v(\{x_j\}) = v_j \quad \text{as} \quad \rho \to 0. \tag{3.11}
\]

Combining (3.7)-(3.11), we obtain

\[
S v_j^{\frac{2}{2^*_s}} \leq \mu_j \leq C v_j^{\frac{2^*_s}{2s}}.
\]

As \( 2^*_s > 2 \) and \( \sum_{j \in J} v_j^{2^*_s} < \infty \), we see that \( v_j \) does not converge to zero, which means that \( J \) is finite. From now on, denote by \( J = \{1, 2, \cdots, m\} \) and \( \Gamma \subset \mathbb{R}^N \) the set given by
\[ \Gamma = \{ x_j \in \{ x_j \} : |x_j| > 1 \}, \quad (x_j \text{ given by (3.5))}. \]

In the sequel, we are going to show that \( v_0 \neq 0 \). Suppose, by contradiction, that \( v_0 = 0 \). Then, by (3.5) we get

\[
\int_{\mathbb{R}^N} |v_n|^{2s} \phi dx \to 0, \quad \forall \phi \in \mathcal{C}^\infty_c (\mathbb{R}^N \setminus \{ x_1, x_2, \ldots, x_m \}).
\]

(3.12)

Since the sequence \( \phi_n(x) = \phi(x)v_n(x) \) with \( \phi \in \mathcal{C}^\infty_c (\mathbb{R}^N \setminus \{ x_1, x_2, \ldots, x_m \}) \), is bounded, we have

\[
I'_{\infty}(v_n)\phi_n = o_n(1),
\]

that is

\[
\begin{align*}
& o_n(1) \\
& = \int \left( \frac{(v_n(x) - v_n(y))(\phi(x)v_n(x) - \phi(y)v_n(y))}{|x - y|^{N+2s}} \right) dxdy - \int (I_{\alpha} * |v_n|^{2s})|v_n|^{2s} \phi \phi_n dx \\
& = \int \left( \frac{v_n(x)(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} \right) dxdy + \int \phi(x)^2 |v_n(x) - v_n(y)|^2 \frac{dxdy}{|x - y|^{N+2s}} \\
& \quad - \int (I_{\alpha} * |v_n|^{2s})|v_n|^{2s} \phi dx.
\end{align*}
\]

(3.13)

Then,

\[
\begin{align*}
& \left| \int \frac{\phi(x)^2 |v_n(x) - v_n(y)|^2}{|x - y|^{N+2s}} dxdy \right| \\
& = \left| \int (I_{\alpha} * |v_n|^{2s})|v_n|^{2s} \phi dx - \int v_n(x) \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy + o_n(1) \right| \\
& \leq \left| \int (I_{\alpha} * |v_n|^{2s})|v_n|^{2s} \phi dx \right| + \left| \int v_n(x) \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy \right| + o_n(1) \\
& \leq \left| \int (I_{\alpha} * |v_n|^{2s})|v_n|^{2s} \phi dx \right| + \left| \int v_n(x) \frac{(v_n(x) - v_n(y))(\phi(x) - \phi(y))}{|x - y|^{N+2s}} dxdy \right| + o_n(1) \\
& \leq \left( \int (I_{\alpha} * |v_n|^{2s})|v_n|^{2s} \phi dx \right) + \|v_n\| \left( \int \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dxdy \right)^{1/2} + o_n(1).
\end{align*}
\]

(3.14)
Now, we show that
\[
\iint_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy = o_n(1). \tag{3.15}
\]

For this, let \( r > 0 \) be such that \( \text{supp}(\phi) \subset B_r(0) \), and write \( \mathbb{R}^{2N} \) as
\[
\mathbb{R}^{2N} = \left[ (\mathbb{R}^N \setminus B_r(0)) \times (\mathbb{R}^N \setminus B_r(0)) \right] \cup \left[ (B_r(0)) \times (\mathbb{R}^N \setminus B_r(0)) \right] \cup \left[ (\mathbb{R}^N \setminus B_r(0)) \times (B_r(0)) \right]
\]
\[= \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.\]

Then we have
\[
\iint_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[= \iint_{\Lambda_1} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy + \iint_{\Lambda_2} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[+ \iint_{\Lambda_3} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy. \tag{3.16}
\]

We are going to estimate each integral in (3.16). Since \( \phi = 0 \) in \( \mathbb{R}^N \setminus B_r(0) \), we have
\[
\iint_{\Lambda_1} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy = 0. \tag{3.17}
\]

Using \( |\phi| \leq C_1, |\nabla \phi| \leq C_2 \) and the mean value theorem, we infer that
\[
\iint_{\Lambda_2} |v_n(x)|^2 \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dx \, dy
\]
\[= \int_{B_r(0)} |v_n(x)|^2 \, dx \iint_{\{y \in \mathbb{R}^N : |x-y| \leq r\}} \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dy
\]
\[+ \int_{B_r(0)} |v_n(x)|^2 \, dx \iint_{\{y \in \mathbb{R}^N : |x-y| > r\}} \frac{\left|\phi(x) - \phi(y)\right|^2}{|x-y|^{N+2s}} \, dy
\]
\[\leq C \|
abla \phi\|_{L^\infty(\mathbb{R}^N)} \int_{B_r(0)} |v_n(x)|^2 \, dx \int_{\{y \in \mathbb{R}^N : |x-y| \leq r\}} \frac{1}{|x-y|^{N+2s-2}} \, dy
\]
\[+ C \int_{B_r(0)} |v_n(x)|^2 \, dx \int_{\{y \in \mathbb{R}^N : |x-y| > r\}} \frac{1}{|x-y|^{N+2s}} \, dy
\]
\[
\int_{B_r(0)} |v_n(x)|^2 \, dx + C r^{-2s} \int_{B_r(0)} |v_n(x)|^2 \, dx = o_n(1). 
\]

(3.18)

Regarding the integral on \( \Lambda_3 \), we have

\[
\int_{\Lambda_3} |v_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dy \\
= \int_{\mathbb{R}^N \setminus B_r(0)} |v_n(x)|^2 \, dx \int_{\{y \in B_r(0): |x - y| \leq r\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dy \\
+ \int_{\mathbb{R}^N \setminus B_r(0)} |v_n(x)|^2 \, dx \int_{\{y \in B_r(0): |x - y| > r\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dy \\
= A_r + B_r. 
\]

(3.19)

Note that if \((x, y) \in (\mathbb{R}^N \setminus B_r(0)) \times B_r(0)\) and \(|x - y| \leq r\), then \(|x| \leq 2r\), we obtain

\[
A_r \leq \int_{\mathbb{R}^N \setminus B_r(0)} |v_n(x)|^2 \, dx \int_{\{y \in B_r(0): |x - y| \leq r\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dy \\
\leq C \|\nabla \phi\|_{L^\infty(\mathbb{R}^N)}^2 \int_{B_{2r}(0)} |v_n(x)|^2 \, dx \int_{\{z \in B_r(0): |z| \leq r\}} \frac{1}{|z|^{N+2s-2}} \, dy \\
= C r^{-2s} \int_{B_{2r}(0)} |v_n(x)|^2 \, dx = o_n(1). 
\]

(3.20)

We observe that, there exists \( K > 4 \) such that

\[
\Lambda_3 = (\mathbb{R}^N \setminus B_r(0)) \times B_r(0) \subset [B_{Kr}(0) \times B_r(0)] \cup [(\mathbb{R}^N \setminus B_{Kr}(0)) \times B_r(0)].
\]

Then, we have the following estimates:

\[
\int_{B_{Kr}(0)} |v_n(x)|^2 \, dx \int_{\{y \in B_r(0): |x - y| > r\}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dy \\
\leq C \int_{B_{Kr}(0)} |v_n(x)|^2 \, dx \int_{\{|z| \in \mathbb{R}^N: |z| > r\}} \frac{1}{|z|^{N+2s}} \, dz \\
= C r^{-2s} \int_{B_{Kr}(0)} |v_n(x)|^2 \, dx = o_n(1). 
\]

(3.21)

We note that if \((x, y) \in (\mathbb{R}^N \setminus B_{Kr}(0)) \times B_r(0)\), then \(|x - y| \geq |x| - |y| \geq \frac{|x|}{2} + \frac{K}{2} r - r > \frac{|x|}{2}\), and by using Hölder’s inequality, we get
\[
\int_{\mathbb{R}^N \setminus B_{Kr}(0)} |v_n(x)|^2 \, dx = \int_{\left\{ y \in B_r(0) : |x-y| > r \right\}} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dy \\
\leq C \int_{\mathbb{R}^N \setminus B_{Kr}(0)} \frac{|v_n(x)|^2}{|x-y|^{N+2s}} \, dy \\
\leq C r^N \int_{\mathbb{R}^N \setminus B_{Kr}(0)} \frac{|v_n(x)|^2}{|x|^{N+2s}} \, dx \\
\leq C r^N \left( \int_{\mathbb{R}^N \setminus B_{Kr}(0)} |v_n(x)|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s}} \left( \int_{\mathbb{R}^N \setminus B_{Kr}(0)} |x|^{-(N+2s)\frac{2^*_s}{2^*_s-2}} \, dx \right)^{\frac{2^*_s-2}{2^*_s}} \\
\leq C K^{-N} \left( \int_{\mathbb{R}^N \setminus B_{Kr}(0)} |v_n(x)|^{2^*_s} \, dx \right)^{\frac{2}{2^*_s}} \leq C_1 K^{-N}. \tag{3.22}
\]

From (3.21) and (3.22), we obtain

\[
B_r \leq C_1 K^{-N} + o_n(1). \tag{3.23}
\]

Combining (3.14)-(3.20) and (3.23), we derive

\[
\limsup_{n \to +\infty} \int \int_{\mathbb{R}^{2N}} |v_n(x)|^2 \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} \, dxdy = 0. \tag{3.24}
\]

By Proposition 2.1 and (3.12), we have

\[
\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2^*_s}) |v_n|^{2^*_s} \phi \, dx \leq C \left| v_n \right|^{2^*_s}_{2^*_s} \left( \int_{\mathbb{R}^N} |v_n|^{2^*_s} \phi \, dx \right)^{\frac{2^*_s}{2^*_s-2}} = o_n(1). \tag{3.25}
\]

From (3.13), (3.24) and (3.25), we can conclude that

\[
\int \int_{\mathbb{R}^{2N}} \phi(y) \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}} \, dxdy \to 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^N \setminus \{x_1, x_2, \ldots, x_m\}), \tag{3.26}
\]

which leads to
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \phi \, dx = o_n(1). \tag{3.27}
\]

Let \( \rho \in \mathbb{R} \) be a number such that \( 0 < \rho < \min \{ \text{dist}(\Gamma, B_1(0)), 1 \} \). We claim that
\[
\int_{B_{1+\rho}(0) \setminus B_{1+\rho 3}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 \, dx \to 0. \tag{3.28}
\]
Indeed, let \( \phi \in C^\infty_c(\mathbb{R}^N) \) be such that \( 0 \leq \phi(x) \leq 1 \) and \( \phi(x) = 1 \) if \( x \in B_{1+\rho}(0) \). Set \( \tilde{\phi} = \phi|_{\mathbb{R}^N \setminus \{x_1, \ldots, x_m\}} \), by (3.27) we get
\[
0 \leq \int_{B_{1+\rho}(0) \setminus B_{1+\rho 3}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 \, dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 \tilde{\phi} \, dx \to 0,
\]
which implies that (3.28) is true.

Let \( \Psi \in C^\infty_c(\mathbb{R}^N) \) be such that \( 0 \leq \Psi(x) \leq 1 \), \( |\nabla \Psi(x)| \leq 2 \) for all \( x \in \mathbb{R}^N \) and
\[
\Psi(x) = \begin{cases} 
1, & x \in B_{1+\rho}(0) \\
0, & x \in B_{1+2\rho}(0) \setminus B_{1+\rho}(0) 
\end{cases}
\]
and consider the sequence \((\Psi_n)\) given by \( \Psi_n(x) = \Psi(x) v_n(x) \).

Now, using (3.15), one has
\[
\int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 \, dx
\]
\[
\leq 2 \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} \int_{\mathbb{R}^N} v_n(y)^2 \frac{|\Psi(x) - \Psi(y)|^2}{|x - y|^{N+2s}} \, dxdy
\]
\[
+ 2 \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} \int_{\mathbb{R}^N} \frac{|\Psi(x)^2| v_n(x) - v_n(y)|^2}{|x - y|^{N+2s}} \, dxdy \tag{3.29}
\]
\[
= o_n(1) + 2 \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} \Psi(x)^2 |(-\Delta)^{\frac{s}{2}} v_n(x)|^2 \, dx
\]
\[
= o_n(1).
\]
Similarly, we can obtain the following estimate.
\[ \int_{B_{1+r}(0) \setminus B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx \]
\[ \leq 2 \int_{B_{1+r}(0) \setminus B_{1+\rho}(0)} \int_{\mathbb{R}^N} v_n(x)^2 |\Psi(x) - \Psi(y)|^2 \frac{dx dy}{|x - y|^{N+2s}} \]
\[ + 2 \int_{B_{1+r}(0) \setminus B_{1+\rho}(0)} \int_{\mathbb{R}^N} |\Psi(y)^2| v_n(x) - v_n(y)^2 \frac{dx dy}{|x - y|^{N+2s}} \]
\[ \leq 2 \int_{B_{1+r}(0)} \int_{\mathbb{R}^N} v_n(x)^2 |\Psi(x) - \Psi(y)|^2 \frac{dx dy}{|x - y|^{N+2s}} \]
\[ + 2 \int_{B_{1+r}(0)} \int_{\mathbb{R}^N} |v_n(x) - v_n(y)|^2 \frac{dx dy}{|x - y|^{N+2s}} \]
\[ = 2 \int_{B_{1+r}(0)} \int_{\mathbb{R}^N} v_n(x)^2 |\Psi(x) - \Psi(y)|^2 \frac{dx dy}{|x - y|^{N+2s}} \]
\[ + 2 \int_{B_{1+r}(0) \setminus B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \]
\[ = o_n(1), \]

where in the last equality we have used the estimates (3.15) and (3.28).

Since \((v_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\), we derive that

\[ \int_{B_{1+r}(0) \setminus B_{1+\rho}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx + \int_{B_{1+\rho}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx \]
\[ - \int_{B_{1+r}(0) \setminus B_{1+\rho}(0)} (I_\alpha * |v_n|^{2s}) |v_n|^{2s} \Psi dx - \int_{B_{1+\rho}(0)} (I_\alpha * |v_n|^{2s}) |v_n|^{2s} \Psi dx \]
\[ = - \int_{\mathbb{R}^N \setminus B_{1+r}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx + \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} (I_\alpha * |v_n|^{2s}) |v_n|^{2s} \Psi dx + o_n(1) \]
\[ = - \int_{\mathbb{R}^N \setminus B_{1+r}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx + o_n(1) \]

Notice that from Hölder’s inequality, (3.29) and (3.30), we get

\[ \int_{\mathbb{R}^N \setminus B_{1+r}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx \to 0 \quad \text{as} \quad n \to \infty, \quad (3.32) \]

and
\[
\int_{B_{1+\rho}(0)\setminus B_{1+\rho/2}(0)} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \Psi_n dx \to 0 \quad \text{as} \quad n \to \infty. \tag{3.33}
\]

Moreover, by Proposition 2.1 and (3.12) we get
\[
\int_{B_{1+\rho}(0)\setminus B_{1+\rho/2}(0)} (I_{\alpha} \ast |v_n|^{2^*_{\alpha,s}})|v_n|^{2^*_{\alpha,s}} \Psi dx = o_n(1). \tag{3.34}
\]

From (3.31)-(3.34) we obtain
\[
\int_{B_{1+\rho}(0)\setminus B_{1+\rho/2}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx - \int_{B_{1+\rho}(0)\setminus B_{1+\rho/2}(0)} (I_{\alpha} \ast |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx = o_n(1). \tag{3.35}
\]

Note that
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx = \int_{\mathbb{R}^N \setminus B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx + \int_{B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx
\]
\[
= \int_{B_{1+\rho}(0)\setminus B_{1+\rho/2}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx + \int_{B_{1+\rho/2}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx + o_n(1)
\]
\[
= o_n(1) + \int_{B_{1+\rho/2}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx \tag{3.36}
\]

and
\[
\int_{\mathbb{R}^N} (I_{\alpha} \ast |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx
\]
\[
= \int_{B_{1+\rho}(0)} (I_{\alpha} \ast |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx \tag{3.37}
\]
\[
= \int_{B_{1+\rho}(0)\setminus B_{1+\rho/2}(0)} (I_{\alpha} \ast |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx + \int_{B_{1+\rho/2}(0)} (I_{\alpha} \ast |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx.
\]

From (3.34), (3.36)-(3.37), we derive that
\[
\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx - \int_{\mathbb{R}^N} (I_{\alpha} \ast |\Psi_n|^{2^*_{\alpha,s}})|\Psi_n|^{2^*_{\alpha,s}} dx = o_n(1).
\]

Recall the definition of $S_{h,l}$, we have
\[ \|\Psi_n\| \geq \left[1 - \frac{1}{S_{h,l}^2} \|\Psi_n\|^2_{H_0^s} - \frac{1}{S_{h,l}^2} \right] = \|\Psi_n\|^2 - \frac{1}{S_{h,l}^2} \|\Psi_n\|^2_{H_0^s} \]
\[ \leq \|\Psi_n\|^2 - \int_{\mathbb{R}^N} (I_{\alpha} * |\Psi_n|^2_{H_0^s})|\Psi_n|^2_{H_0^s} \, dx = o_n(1). \tag{3.38} \]

Since
\[ \|\Psi_n\|^2 = \int_{B_{1+\rho}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 \, dx + \int_{B_{1+\rho}(0) \setminus B_{1+\frac{\rho}{2}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 \, dx + \int_{B_{1+\frac{\rho}{2}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 \, dx \]
\[ = o_n(1) + \int_{B_{1+\frac{\rho}{2}}(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 \, dx \]
\[ \leq o_n(1) + \int_{B_2} |(-\Delta)^{\frac{s}{2}} v_n|^2 \, dx, \]
where in the last inequality we have used \( \Psi_n = v_n \) in \( B_{1+\frac{\rho}{2}}(0) \) and that \( B_{1+\frac{\rho}{2}}(0) \subset B_2(0) \). Therefore, we arrive at
\[ \|\Psi_n\|^2 \leq o_n(1) + \int_{\bigcup_{k=1}^L B_1(y_k)} |(-\Delta)^{\frac{s}{2}} v_n|^2 \, dx \]
\[ \leq o_n(1) + \sum_{k=1}^L \int_{B_1(y_k)} |(-\Delta)^{\frac{s}{2}} v_n|^2 \, dx \]
\[ \leq o_n(1) + L \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |(-\Delta)^{\frac{s}{2}} v_n|^2 \, dx \leq o_n(1) + \frac{\|v_n\|_{\mathbb{R}^N}^{2(N-s)/(N+2s-\alpha)}}{2}, \]
which gives that
\[ \|\Psi_n\| \leq o_n(1) + \frac{\|v_n\|_{\mathbb{R}^N}^{2(N-s)/(N+2s-\alpha)}}{2^{1/2}} \Rightarrow \|\Psi_n\|_{H_0^s}^{22_{a,s}^2} \leq o_n(1) + \left( \frac{\|v_n\|_{\mathbb{R}^N}^{2(N-s)/(N+2s-\alpha)}}{2^{1/2}} \right)^{22_{a,s}^2} \]
\[ \Rightarrow o_n(1) - \left( \frac{\|v_n\|_{\mathbb{R}^N}^{2(N-s)/(N+2s-\alpha)}}{2^{1/2}} \right)^{22_{a,s}^2} \leq -\|\Psi_n\|_{H_0^s}^{22_{a,s}^2}. \tag{3.39} \]

It follows from (3.38), (3.39) that
\[ \| \Psi_n \|^2 \left[ 1 + o_n(1) - \frac{1}{S_{h,l}^{2s}} \left( \frac{S_{h,l}^{(2N-\alpha)/(2(N+2s-\alpha))}}{2^{1/2}} \right)^{22s_{\alpha,s}-2} \right] \]
\[ = \| \Psi_n \|^2 \left[ 1 + \frac{1}{S_{h,l}^{2s}} \left[ o_n(1) - \left( \frac{S_{h,l}^{(2N-\alpha)/(2(N+2s-\alpha))}}{2^{1/2}} \right)^{22s_{\alpha,s}-2} \right] \right] \]  
\[ \leq \| \Psi_n \|^2 \left[ 1 - \frac{1}{S_{h,l}^{2s}} \| \Psi_n \|^{22s_{\alpha,s}-2} \right] \leq o_n(1). \]  

Let us observe that

\[ \frac{2N - \alpha}{2(N + 2s - \alpha)} (22s_{\alpha,s} - 2) - 2s_{\alpha,s} \]
\[ = \frac{2N - \alpha}{2(N + 2s - \alpha)} \times \left[ \frac{2(2N - \alpha)}{N - 2s} - 2 \right] - \frac{2N - \alpha}{N - 2s} \]
\[ = 0, \]

implies

\[ \| \Psi_n \|^2 \left[ 1 - \left( \frac{1}{2} \right)^{2s_{\alpha,s}-1} \right] \leq o_n(1), \]

where we conclude that \( \Psi_n \to 0 \) in \( D^{s,2}(\mathbb{R}^N) \). Since \( v_n = \Psi_n \) in \( B_1(0) \), we infer that

\[ 0 \leq \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx = \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_n|^2 dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx, \]

which yields to

\[ \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx \to 0 \quad \text{as} \quad n \to \infty. \]

But this last convergence contradicts to

\[ \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx = \frac{S_{h,l}^{N-\alpha}}{2L}, \quad \forall n \in \mathbb{N}. \]

Hence, \( v_0 \neq 0 \).

Now, we are going to show that there is \( (w_n) \) in \( D^{s,2}(\mathbb{R}^N) \) such that \( (w_n) \) is a \( (P.S)_{c-v_0} \) sequence for \( I_{\infty} \) satisfying

\[ w_n(x) = u_n(x) - R_n^{(N-2s)/2} v_0(R_n(x-x_n)) + o_n(1), \]

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for some subsequence of \((u_n)\) that still denoted by \((u_n)\). For this, we consider \(\psi \in C^\infty_0(\mathbb{R}^N)\) such that \(0 \leq \psi \leq 1\) for all \(x \in \mathbb{R}^N\) and

\[
\psi(x) = \begin{cases} 
1, & \text{if } x \in B_1(0), \\
0, & \text{if } x \in B_2^c(0), 
\end{cases}
\]

and define a sequence by

\[
u_n(x) = u_n(x) - R_n^{(N-2s)/2} v_0(R_n(x-x_n)) \psi(R_n(x-x_n)),
\]

(3.41)

where \((R_n)\) satisfies \(\tilde{R}_n = \frac{R_n}{R_n} \to \infty\). From (3.41) we get

\[
R_n^{(2s-N)/2} w_n(x) = R_n^{(2s-N)/2} u_n(x) - v_0(R_n(x-x_n)) \psi(R_n(x-x_n)).
\]

Making change of variable, we get

\[
R_n^{(2s-N)/2} w_n \left( \frac{z}{R_n} + x_n \right) = R_n^{(2s-N)/2} u_n \left( \frac{z}{R_n} + x_n \right) - v_0(z) \psi \left( \frac{z}{R_n} \right).
\]

Denote by

\[
\tilde{w}_n(z) = R_n^{(2s-N)/2} w_n \left( \frac{z}{R_n} + x_n \right)
\]

and since

\[
u_n(x) = R_n^{(2s-N)/2} u_n \left( \frac{z}{R_n} + x_n \right),
\]

we have,

\[
\tilde{w}_n(z) = v_n(z) - v_0(z) \psi \left( \frac{z}{R_n} \right).
\]

(3.42)

Set

\[
\psi_n(z) := \psi \left( \frac{z}{R_n} \right),
\]

(3.43)

we see that

\[
\psi_n(z) = \begin{cases} 
1, & \text{if } z \in B_{\tilde{R}_n}(0), \\
0, & \text{if } z \in B_{2R_n}^c(0).
\end{cases}
\]

By (3.42) and (3.43) we derive that

\[
\tilde{w}_n(z) = v_n(z) - v_0(z) \psi_n(z).
\]
The proof is over if we can show that $v_0 \psi_n \to v_0$ in $D^{s,2}(\mathbb{R}^N)$ and so $(w_n)$ is a $(PS)_{c-I_{\infty}(v_0)}$ sequence for $I_{\infty}$. To this aim, we note that

\[
\|v_0\psi_n - v_0\|^2
\]

\[
= \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} (v_0\psi_n - v_0)|^2\,dx
\]

\[
= \int_{\mathbb{R}^{2N}} \frac{|v_0(x)\psi_n(x) - v_0(x) - v_0(y)\psi_n(y) + v_0(y)|^2}{|x - y|^{N+2s}}\,dxdy
\]

\[
= \int_{\mathbb{R}^{2N}} \frac{|v_0(x)[\psi_n(x) - \psi_n(y)] + [\psi_n(y) - 1][v_0(x) - v_0(y)]|^2}{|x - y|^{N+2s}}\,dxdy
\]

\[
\leq 2 \int_{\mathbb{R}^{2N}} \frac{|v_0(x)|^2|\psi_n(x) - \psi_n(y)|^2}{|x - y|^{N+2s}}\,dxdy + 2 \int_{\mathbb{R}^{2N}} \frac{|\psi_n(y) - 1|^2|v_0(x) - v_0(y)|^2}{|x - y|^{N+2s}}\,dxdy
\]

\[
(3.44)
\]

As in the proof of (3.15), if we replace $v_n$ by $v_0$, and $\phi$ by $\psi_n$, since $\text{supp}(\psi_n) \subset B_{2\bar{\rho}}(0)$, it is easy to obtain the estimate

\[
\int_{\mathbb{R}^{2N}} \frac{|v_0(x)|^2|\psi_n(x) - \psi_n(y)|^2}{|x - y|^{N+2s}}\,dxdy = o_n(1).
\]

On the other hand, taking into account that $|\psi_n(x) - 1| \leq 2$, $|\psi_n(x) - 1| \to 0$ a.e. in $\mathbb{R}^N$ and $v_0 \in D^{s,2}(\mathbb{R}^N)$, the Dominated Convergence Theorem implies that

\[
\int_{\mathbb{R}^{2N}} \frac{|\psi_n(y) - 1|^2|v_0(x) - v_0(y)|^2}{|x - y|^{N+2s}}\,dxdy \to 0 \quad \text{as} \quad n \to \infty.
\]

Coming back to (3.44) we get

\[
v_0\psi_n \to v_0 \quad \text{in} \quad D^{s,2}(\mathbb{R}^N)
\]

which implies that,

\[
\tilde{w}_n = v_n - v_0 + o_n(1).
\]

Since $v_n \to v_0$ in $D^{s,2}(\mathbb{R}^N)$ and $v_n \to v_0$ a.e. in $\mathbb{R}^N$, we have

\[
\int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} v_n|^2\,dx = \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} v_0|^2\,dx + \int_{\mathbb{R}^N} |(\Delta)^{\frac{s}{2}} (v_n - v_0)|^2\,dx + o_n(1),
\]

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\[
\int_{\mathbb{R}^N} (I_\alpha * |v_n|^{2s}) |v_n|^{2s} d\mathbf{x} = \int_{\mathbb{R}^N} (I_\alpha * |v_0|^{2s}) |v_0|^{2s} d\mathbf{x} + \int_{\mathbb{R}^N} (I_\alpha * |v_n - v_0|^{2s}) |v_n - v_0|^{2s} d\mathbf{x} + o_n(1),
\]

which implies that

\[
I_\infty(w_n) = \int_{\mathbb{R}^N} \frac{1}{2} \Delta^s \tilde{w}_n - \frac{1}{2} \Delta^s \tilde{w}_n d\mathbf{x} - \frac{1}{2} \Delta^s \tilde{v}_n - \frac{1}{2} \Delta^s \tilde{v}_n d\mathbf{x} + o_n(1)
\]

\[
= I_\infty(v_n) - I_\infty(v_0) + o_n(1).
\]

Therefore,

\[
I_\infty(w_n) \rightarrow \tilde{c} \quad \text{as} \quad n \rightarrow \infty,
\]

where \(\tilde{c} = c - I_\infty(v_0)\). Now, we are going to show that

\[
\|I'_\infty(\tilde{w}_n) - I'_\infty(v_n) + I'_\infty(v_0)\|_{D^{-s,2} (\mathbb{R}^N)} \rightarrow 0. \tag{3.45}
\]

Indeed, for all \(\Phi \in D^{s,2} (\mathbb{R}^N)\) with \(\|\Phi\| \leq 1\), we have

\[
\left| \left( I'_\infty(\tilde{w}_n) - I'_\infty(v_n) + I'_\infty(v_0) \right) \Phi \right|
\]

\[
= \left| \int_{\mathbb{R}^N} (\Delta^s \tilde{w}_n - \Delta^s \tilde{v}_n - \Delta^s v_0) \Phi d\mathbf{x} - \int_{\mathbb{R}^N} (I_\alpha * |\tilde{w}_n|^{2s}) |\tilde{w}_n|^{2s} - \int_{\mathbb{R}^N} (I_\alpha * |\tilde{v}_n|^{2s}) |\tilde{v}_n|^{2s} - \int_{\mathbb{R}^N} (I_\alpha * |v_0|^{2s}) |v_0|^{2s} d\mathbf{x} \right|
\]

\[
= \left| \int_{\mathbb{R}^N} \left[ (\Delta^s \tilde{w}_n - \Delta^s \tilde{v}_n - \Delta^s v_0) \Phi d\mathbf{x} \right] \right|
\]

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\[ - \int_{\mathbb{R}^N} \left[ (I_\alpha * |\tilde{w}_n|^{2_{\alpha,s}})|\tilde{w}_n|^{2_{\alpha,s}-2}\tilde{w}_n - (I_\alpha * |v_n|^{2_{\alpha,s}})|v_n|^{2_{\alpha,s}-2}v_n \\
\quad + (I_\alpha * |v_0|^{2_{\alpha,s}})|v_0|^{2_{\alpha,s}-2}v_0 \right] \Phi dx \]
\[ \leq \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} \tilde{w}_n - (-\Delta)^{\frac{s}{2}} v_n + (-\Delta)^{\frac{s}{2}} v_0 \right|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} \Phi \right|^2 dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N} \left[ (I_\alpha * |v_n|^{2_{\alpha,s}})|v_n|^{2_{\alpha,s}-2}v_n - (I_\alpha * |\tilde{w}_n|^{2_{\alpha,s}})|\tilde{w}_n|^{2_{\alpha,s}-2}\tilde{w}_n \right. \right. \\
- \left. \left. (I_\alpha * |v_0|^{2_{\alpha,s}})|v_0|^{2_{\alpha,s}-2}v_0 \right] \frac{2^*_s}{2^*_s - 1} dx \right)^{\frac{2^*_s - 1}{2^*_s}} \left( \int_{\mathbb{R}^N} |\Phi|^{2^*_s} dx \right)^{\frac{1}{2^*_s}}. \quad (3.46) \]

By the definition of \( \tilde{w}_n \), we have
\[ \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} \tilde{w}_n - (-\Delta)^{\frac{s}{2}} v_n + (-\Delta)^{\frac{s}{2}} v_0 \right|^2 dx = o_n(1). \quad (3.47) \]

Using Lemma \textit{A A}, we get
\[ \int_{\mathbb{R}^N} \left[ (I_\alpha * |v_n|^{2_{\alpha,s}})|v_n|^{2_{\alpha,s}-2}v_n - (I_\alpha * |\tilde{w}_n|^{2_{\alpha,s}})|\tilde{w}_n|^{2_{\alpha,s}-2}\tilde{w}_n \right. \]
\[ - \left. (I_\alpha * |v_0|^{2_{\alpha,s}})|v_0|^{2_{\alpha,s}-2}v_0 \right] \frac{2^*_s}{2^*_s - 1} dx = o_n(1). \]

Since \( v_0 \) is a nontrivial critical point of \( I_\infty \), we conclude that
\[ I_\infty'(\tilde{w}_n) = I_\infty'(v_n) + o_n(1) = I_\infty'(v_0) + o_n(1) = o_n(1). \]

As \( \| I_\infty'(w_n) \|_{D^{-1,2}(\mathbb{R}^N)} \leq \| I_\infty'(\tilde{w}_n) \|_{D^{-1,2}(\mathbb{R}^N)} \), it follows that \( I_\infty'(w_n) \to 0 \) and the proof of this lemma is over. \( \Box \)

The next result is a version of nonlocal global compactness result for fractional Laplacian Choquard equation in \( \mathbb{R}^N \) with critical growth.

**Theorem 3.2.** (A splitting theorem) Let \( (u_n) \) be a \((P.S)_c\) sequence for \( I \) with \( u_n \to u_0 \) in \( D^{s,2}(\mathbb{R}^N) \). Then, up to a subsequence, \( (u_n) \) satisfies either,

(a) \( u_n \to u_0 \) in \( D^{s,2}(\mathbb{R}^N) \) or,
(b) there exists $k \in \mathbb{N}$ and nontrivial solutions $z^1_0, z^2_0, \ldots, z^k_0$ for problem $(P)_{\infty}$, such that

$$\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^k \|z^j_0\|^2$$

and

$$I(u_n) \to I(u_0) + \sum_{j=1}^k I_{\infty}(z^j_0).$$

Proof. From the weak convergence, we have that $u_0$ is a critical point of $I$. Assume that $u_n \nrightarrow u_0$ in $D^{s,2}(\mathbb{R}^N)$ and let $(z^1_n) \subset D^{s,2}(\mathbb{R}^N)$ be the sequence given by $z^1_n := u_n - u_0$. Then by hypothesis, we have $z^1_n \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ and $z^1_n \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$.

Using Lemma 3 [2] and Lemma A, we have

$$I_{\infty}(z^1_n) = I(u_n) - I(u_0) + o_n(1) \quad (3.48)$$

and

$$I'_{\infty}(z^1_n) = I'(u_n) - I'(u_0) + o_n(1). \quad (3.49)$$

By (3.48) and (3.49) we see that $(z^1_n)$ is a $(PS)_{c_1}$ sequence for $I_{\infty}$. Hence, by Lemma 3.1, there exist sequences $(R_n, 1) \subset \mathbb{R}$, $(x_n, 1) \subset \mathbb{R}^N$, $z^1_0 \in D^{s,2}(\mathbb{R}^N)$ nontrivial solution of $(P_{\infty})$ and a $(PS)_{c_2}$ sequence $(z^2_n)$ for $I_{\infty}$ such that

$$z^2_n(x) = z^1_n(x) - R_{n,1}^{(N-2s)/2} z^1_0(R_{n,1}(x - x_n, 1)) + o_n(1).$$

If we define

$$v^1_n(x) = R_{n,1}^{(2s-N)/2} z^1_n \left( \frac{x}{R_{n,1}} + x_n, 1 \right) \quad (3.50)$$

and

$$\tilde{z}^2_n(x) = R_{n,1}^{(2s-N)/2} z^2_n \left( \frac{x}{R_{n,1}} + x_n, 1 \right),$$

we get

$$\tilde{z}^2_n(x) = v^1_n(x) - z^1_0(x) + o_n(1) \quad (3.51)$$

and

$$\|v^1_n\| = \|z^1_n\| \quad \text{and} \quad \int_{\mathbb{R}^N} (I_{\alpha} * |v^1_n|^2_{2,s}) |v^1_n|^{2s}_{2,s} dx = \int_{\mathbb{R}^N} (I_{\alpha} * |z^1_n|^2_{2,s}) |z^1_n|^{2s}_{2,s} dx. \quad (3.52)$$
Hence,
\[ I_\infty(v_1^n) = I_\infty(z_1^n) \quad \text{and} \quad I'_\infty(v_1^n) \to 0 \quad \text{in} \quad D^{-s,2}(\mathbb{R}^N). \]  
(3.53)

From (3.53) and Lemma 2.2, we have that \((v_1^n)\) is a bounded sequence in \(D^{s,2}(\mathbb{R}^N)\) and, up to a subsequence,
\[ v_1^n \rightharpoonup z_0^1 \quad \text{in} \quad D^{s,2}(\mathbb{R}^N). \]  
(3.54)

As we arguing before, we conclude
\[ I_\infty(\tilde{z}_2^n) = I_\infty(v_1^n) - I_\infty(z_0^1) + o_n(1) = I_\infty(u_n) - I_\infty(u_0) - I_\infty(z_0^1) + o_n(1), \]  
(3.55)

and
\[ I'_\infty(\tilde{z}_2^n) = I'_\infty(v_1^n) - I'_\infty(z_0^1) + o_n(1). \]  
(3.56)

If \(\tilde{z}_2^n \to 0\) in \(D^{s,2}(\mathbb{R}^N)\), the proof is over for \(k = 1\), since in this case, we get
\[ \|u_n\|_2^2 \to \|u_0\|^2 + \|z_0^1\|^2. \]

If \(\tilde{z}_2^n \rightharpoonup 0\), using (3.51) and (3.54) that \(\tilde{z}_2^n \rightharpoonup 0\), by (3.55) and (3.56) that \((\tilde{z}_2^n)\) is a \((PS)_{c_2}\) sequence for \(I_\infty\). By Lemma 3.1, there exist sequences \((R_{n,2}) \subset \mathbb{R}, (x_{n,2}) \subset \mathbb{R}^N, z_0^2 \in D^{s,2}(\mathbb{R}^N)\) nontrivial solution of \((P_\infty)\) and a \((PS)_{c_3}\) sequence \((z_3^n)\) for \(I_\infty\) such that
\[ z_3^n(x) = \tilde{z}_2^n(x) - R_{n,2}^{(N-2s)/2} z_0^2 (R_{n,2}(x - x_{n,2})) + o_n(1). \]

If we define
\[ v_2^n(x) = R_{n,2}^{(2s-N)/2} z_0^2 \left( \frac{x}{R_{n,2}} + x_{n,2} \right) \]  
(3.57)

and
\[ \tilde{z}_3^n(x) = R_{n,2}^{(2s-N)/2} z_3^n \left( \frac{x}{R_{n,2}} + x_{n,2} \right), \]

we get
\[ \tilde{z}_3^n(x) = v_2^n(x) - z_0^2(x) + o_n(1). \]  
(3.58)

Arguing of same way that was before, we have
\[ \|z_3^n\|^2 = \|u_n\|^2 - \|u_0\|^2 - \|z_0^1\|^2 - \|z_0^2\|^2 + o_n(1), \]  
(3.59)
\[ I_\infty(z_3^n) = I(u_n) - I(u_0) - I_\infty(z_0^1) - I_\infty(z_0^2) + o_n(1), \]  
(3.60)
\[ \begin{align*}
I_\infty'(\tilde{z}_n^3) &= I_\infty'(v_n^2) - I_\infty'(z_0^2) + o_n(1). \\
\text{If } \tilde{z}_n^3 \to 0 \text{ in } D^{2,2}(\mathbb{R}^N), \text{ the proof is over with } k = 2, \text{ because } \|\tilde{z}_n^3\|^2 \to 0, \text{ and by (3.59), we obtain}
\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{2} \|z_0^j\|^2.
\end{align*} \]

Moreover, from continuity of \(I_\infty\), we also have that \(I_\infty(\tilde{z}_n) \to 0\). Thus, by (3.60),
\[ \begin{align*}
I_\infty(u_n) \to I(u_0) + \sum_{j=1}^{2} I_\infty(z_0^j) + o_n(1).
\end{align*} \]

If \(\tilde{z}_n^3 \not\to 0\), we can repeat the same arguments before and we shall find \(z_0^1, z_0^2, \cdots, z_0^{k-1}\) non-trivial solutions of problem \((P_\infty)\) satisfying
\[ \begin{align*}
\|\tilde{z}_n^k\|^2 &= \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k-1} \|z_0^j\|^2 + o_n(1) \quad (3.62)
\end{align*} \]
and
\[ \begin{align*}
I_\infty(\tilde{z}_n^k) &= I(u_n) - I(u_0) - \sum_{j=1}^{k-1} I_\infty(z_0^j) + o_n(1). \quad (3.63)
\end{align*} \]

By the definition of \(S_{h,l}\),
\[ \begin{align*}
S_{h,l} \left( \int_{\mathbb{R}^N} (I_\alpha * |z_0^j|^{2^*_{\alpha,s}}) |z_0^j|^{2^*_{\alpha,s}} \ dx \right)^{\frac{1}{2^*_{\alpha,s}}} \leq \|z_0^j\|^2, \quad j = 1, 2, \cdots, k - 1. \quad (3.64)
\end{align*} \]

Since \(z_0^j\) is nontrivial solution of \((P_\infty)\), for all \(j = 1, 2, \cdots, k - 1\), we obtain
\[ \begin{align*}
\|z_0^j\|^2 &= \int_{\mathbb{R}^N} (I_\alpha * |z_0^j|^{2^*_{\alpha,s}}) |z_0^j|^{2^*_{\alpha,s}} \ dx.
\end{align*} \]

Therefore,
\[ \begin{align*}
\|z_0^j\|^2 \geq S_{h,l}^{\frac{2N-\alpha}{N+2\alpha-\alpha}}, \quad j = 1, 2, \cdots, k - 1. \quad (3.65)
\end{align*} \]

From (3.62) and (3.65),
\[ \|z_n^k\|^2 = \|u_n\|^2 - \|u_0\|^2 - \sum_{j=1}^{k-1} \|z_0^j\|^2 + o_n(1) \]
\[ \leq \|u_n\|^2 - \|u_0\|^2 - (k - 1) S_{h,l}^{2N-\alpha} + o_n(1). \]

Since \((u_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\), for \(k\) large enough, we conclude that \(z_n^k \to 0\) in \(D^{s,2}(\mathbb{R}^N)\) and the proof is completed. \(\square\)

An immediate consequence of the last theorem, are the next two conclusions.

**Corollary 3.3.** Let \((u_n)\) be a \((P S)_c\) sequence for \(I\) with \(c \in \left(0, \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha}\right)\). Then, up to a subsequence, \((u_n)\) converges strongly in \(D^{s,2}(\mathbb{R}^N)\).

**Proof.** We have that \((u_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\), \(u_n \rightharpoonup u_0\) in \(D^{s,2}(\mathbb{R}^N)\) and \(I'(u_0) = 0\). Suppose, by contradiction, that \(u_n \not\rightharpoonup u_0\) in \(D^{s,2}(\mathbb{R}^N)\). From Theorem 3.2, there are \(k \in \mathbb{N}\) and nontrivial solutions \(z_0^1, z_0^2, \ldots, z_0^k\) to problem \((P_\infty)\) such that,

\[ \|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^{k} \|z_0^j\|^2 \]

and

\[ I(u_n) \to I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j). \]

Notice that

\[ I(u_0) = \frac{1}{2} \|u_0\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} a(x)|u_0|^2 dx - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_{\alpha} * |u_0|^{2^*_s}) |u_0|^{2^*_s} dx \]

\[ = \frac{1}{2} \|u_0\|^2 + \frac{1}{2} \left( \int_{\mathbb{R}^N} (I_{\alpha} * |u_0|^{2^*_s}) |u_0|^{2^*_s} dx - \|u_0\|^2 \right) - \frac{1}{22^*_{\alpha,s}} \int_{\mathbb{R}^N} (I_{\alpha} * |u_0|^{2^*_s}) |u_0|^{2^*_s} dx \]

\[ = \left( \frac{1}{2} - \frac{1}{22^*_{\alpha,s}} \right) \int_{\mathbb{R}^N} (I_{\alpha} * |u_0|^{2^*_s}) |u_0|^{2^*_s} dx \geq 0. \]

Therefore,

\[ c = I(u_0) + \sum_{j=1}^{k} I_\infty(z_0^j) \geq \sum_{j=1}^{k} I_\infty(z_0^j) \]

\[ \geq k \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha} \frac{2N-\alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha} > \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha}, \]

which is a contradiction with \(c \in \left(0, \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha}\right)\). \(\square\)

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**Corollary 3.4.** The functional \( I : D^{s,2}(\mathbb{R}^N) \to \mathbb{R} \) satisfies the \((PS)_c\) condition with \( c \in \left( \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{(2N-\alpha)/(N+2s-\alpha)}, \frac{2^s}{4N-2\alpha} \right) \).

**Proof.** Let \((u_n)\) be a \((PS)_c\) sequence in \( D^{s,2}(\mathbb{R}^N) \), that is,

\[
I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0.
\]

Since \((u_n)\) is bounded in \( D^{s,2}(\mathbb{R}^N) \), up to a subsequence, we have \( u_n \rightharpoonup u_0 \) in \( D^{s,2}(\mathbb{R}^N) \) and \( I'(u_0) = 0 \). Suppose, by contradiction, that \( u_n \nrightarrow u_0 \) in \( D^{s,2}(\mathbb{R}^N) \). Then from Theorem 3.2, there are \( k \in \mathbb{N} \) and nontrivial solutions \( z_0^1, z_0^2, \ldots, z_0^k \) to problem \((P_\infty)\) such that,

\[
\|u_n\|^2 \to \|u_0\|^2 + \sum_{j=1}^k \|z_0^j\|^2
\]

and

\[
I(u_n) \to I(u_0) + \sum_{j=1}^k I_\infty(z_0^j).
\]

The above information ensures that \( u_0 \neq 0 \). Since \( I(u_0) \geq 0 \), then \( k = 1 \) and \( z_0^1 \) cannot change its sign, because otherwise, by Lemma 2.3,

\[
I_\infty(z_0^1) \geq 2 \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},
\]

which leads to a contradiction. Hence,

\[
c = I(u_0) + I_\infty(z_0^1).
\]

On the other hand, if \( z_0^1 \) has definite sign, by (2.8) and (2.9), we have

\[
I_\infty(z_0^1) = \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.
\]

On the other hand, by a direct calculation,

\[
I(u_0) \geq \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}}.
\]

Hence,

\[
c = I(u_0) + I_\infty(z_0^1) \geq 2 \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}} > 2 \frac{4^s}{4N-2\alpha} \frac{N+2s-\alpha}{4N-2\alpha} S_{h,l}^{\frac{2N-\alpha}{N+2s-\alpha}},
\]

which yields a contradiction. This completes the proof. \( \square \)
In the sequel, we consider the functional \( f : D^{s,2}(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
    f(u) = \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} + a(x)|u|^2 \right) dx
\]

and the manifold \( \mathcal{M} \subset D^{s,2}(\mathbb{R}^N) \) given by

\[
    \mathcal{M} = \left\{ u \in D^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (I_\alpha * |u|^{2\alpha}) |u|^{2\alpha} dx = 1 \right\}.
\]

The next results are direct consequence of the Corollaries above.

**Lemma 3.5.** Let \((u_n)\) be a sequence that satisfies

\[
    f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.
\]

Then, the sequence \((v_n) \subset D^{s,2}(\mathbb{R}^N)\) with

\[
    v_n = c^{(N-2\alpha)/(2N+4s-2\alpha)} u_n
\]

satisfies the following limits.

\[
    I(v_n) \to \frac{N + 2s - \alpha}{4N - 2\alpha} c^{\frac{2N-\alpha}{2N+4s-2\alpha}} \quad \text{and} \quad I'(v_n) \to 0.
\]

**Lemma 3.6.** Suppose that there are a sequence \((u_n) \subset \mathcal{M}\) and

\[
    c \in \left( S_{h,l}, 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l} \right)
\]

such that

\[
    f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.
\]

Then, up to a subsequence, \(u_n \to u\) in \(D^{s,2}(\mathbb{R}^N)\).

**Corollary 3.7.** Suppose that there is a sequence \((u_n) \subset \mathcal{M}\) and

\[
    c \in \left( S_{h,l}, 2^{\frac{4s-\alpha}{2N-\alpha}} S_{h,l} \right)
\]

such that

\[
    f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0.
\]

Then \(I\) has a critical point \(v_0 \in D^{s,2}(\mathbb{R}^N)\) with

\[
    I(v_0) = \frac{N + 2s - \alpha}{4N - 2\alpha} c^{(2N-\alpha)/(N+2s-\alpha)}.
\]
4. Technical lemmas

Throughout this section, we consider the function $\Psi_{\delta, b} \in D^{s, 2}(\mathbb{R}^N)$ given by

$$\Psi_{\epsilon, b}(x) = S_{h, l} \frac{N-2s}{N+4s-2a} \widetilde{U}_{\epsilon, b}(x).$$

$$= S_{h, l} \frac{N-2s}{N+4s-2a} S_{\frac{(N-\alpha)(2s-N)}{2(N+4s-2a)}} D_{\frac{2s-N}{N+4s-2a}} U_{\epsilon, b}(x),$$

$$= S_{h, l} \frac{N-2s}{N+4s-2a} b_{\alpha, s} \left( \frac{\epsilon}{\epsilon^2 + |x-b|^2} \right)^{\frac{N-2s}{2}}, \quad x, b \in \mathbb{R}^N \text{ and } \epsilon > 0,$$

where $U_{\epsilon, b}(x)$ solves problem $(P_\infty)$ as we have pointed in Section 2. By a simple calculation, we get

$$\|\Psi_{\epsilon, b}\| = \left( \int_{\mathbb{R}^N} (I_{\alpha} * |\Psi_{\epsilon, b}|^{2^*_s}_{\alpha, s}) |\Psi_{\epsilon, b}|^{2^*_s}_{\alpha, s} dx \right)^{\frac{1}{2^*_s}_{\alpha, s}} = 1$$

(4.1)

Now, we prove some properties on the function $\Psi_{\epsilon, b}$ given by in (4.1). First of all, we recall

$$\Psi_{\epsilon, b} \in \Sigma = \{ u \in D^{s, 2}(\mathbb{R}^N) : u \geq 0 \}$$

(4.2)

and

$$\Psi_{\epsilon, b} \in L^q(\mathbb{R}^N) \text{ for } q \in \left[ \frac{N}{N-2s}, \frac{2^*_s}_{\alpha, s} \right], \forall \epsilon > 0 \text{ and } \forall b \in \mathbb{R}^N.$$  

(4.3)

Lemma 4.1. For each $b \in \mathbb{R}^N$, we have

(i) $\|\Psi_{\epsilon, b}\|_{H^{1, \infty}(\mathbb{R}^N)} \to 0$ as $\epsilon \to +\infty$,

(ii) $\|\Psi_{\epsilon, b}\|_{H^{1, \infty}(\mathbb{R}^N)} \to +\infty$ as $\epsilon \to 0$,

(iii) $|\Psi_{\epsilon, b}|_{q} \to 0$ when $\epsilon \to 0$, \forall $q \in \left( \frac{N}{N-2s}, \frac{2^*_s}_{\alpha, s} \right)$,

(iv) $|\Psi_{\epsilon, b}|_{q} \to +\infty$ when $\epsilon \to +\infty$, \forall $q \in \left( \frac{N}{N-2s}, \frac{2^*_s}_{\alpha, s} \right)$.

Proof. Using the definition of $\Psi_{\epsilon, b}$ we have

$$|\nabla \Psi_{\epsilon, b}(x)| = \frac{B_{\alpha, s} \epsilon^{\frac{N-2s}{2}} |x-b|}{[\epsilon^2 + |x-b|^2]^{\frac{N-2s}{2}+1}},$$

where $B_{\alpha, s} > 0$ is a constant given in (4.1), and then

$$\|\Psi_{\epsilon, b}\|_{H^{1, \infty}(\mathbb{R}^N)} = B_{\alpha, s} \epsilon^{-\frac{N+2-2s}{2}}$$
and consequently (i) and (ii) follow. Now
\[ |\Psi_{\varepsilon,b}|^q = B_{\alpha,s}^{q_\varepsilon} e^{N\frac{-2s}{\alpha}} \int_{\mathbb{R}^N} \left( \frac{1}{1+|z|^2} \right)^{N-2s} dz, \]  
(4.6)
and using the hypothesis that \( \frac{N}{N-2s} < q < 2^*_{\alpha,s} < 2^*_{s} \), (iii) and (iv) follow immediately. \( \Box \)

**Lemma 4.2.** For each \( r > 0 \), we have
\[ \int_{\mathbb{R}^N \setminus B_r(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \to 0 \text{ when } \varepsilon \to 0. \]

**Proof.** From Proposition 2.2 in [39], we have
\[ \int_{\mathbb{R}^N \setminus B_r(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \leq C \int_{\mathbb{R}^N \setminus B_r(0)} |\nabla \Psi_{\varepsilon,0}|^2 dx. \]  
(4.7)
Using (4.5) we get
\[ \int_{\mathbb{R}^N \setminus B_r(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,0}|^2 dx \leq C_1 \int_{\mathbb{R}^N \setminus B_r(0)} \frac{\varepsilon^{N-2s} |x|^2}{[\varepsilon^2 + |x|^2]^{N-2s+2}} \leq C_2 \varepsilon^{N-2s-r-N-2+4s} \to 0 \text{ as } \varepsilon \to 0, \]  
(4.8)
which implies conclusion. \( \Box \)

**Lemma 4.3.** Suppose that \( a \in L^q(\mathbb{R}^N) \), \( \forall q \in [p_1, p_2] \), where \( 1 < p_1 < \frac{2N-a}{4s-a} < p_2 < \frac{N}{4s-N} \) if \( 2s < N < 4s \). Then for each \( \tau > 0 \), there are \( \underline{\varepsilon} = \varepsilon(\tau) > 0 \) and \( \overline{\varepsilon} = \overline{\varepsilon}(\tau) > 0 \) such that
\[ \sup_{b \in \mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + \tau, \quad \varepsilon \in [0, \underline{\varepsilon}] \cup [\overline{\varepsilon}, \infty). \]

**Proof.** Let \( y \in \mathbb{R}^N, q \in \left[ \frac{2N-a}{4s-a}, p_2 \right] \) and \( t \in (1, \infty) \) with \( \frac{1}{q} + \frac{1}{t} = 1 \). By a simple calculus,
\[ \frac{N}{N-2s} < 2t < 2^*_{\alpha,s}. \]  
(4.9)
Since \( \Psi_{\varepsilon,b} \in L^d(\mathbb{R}^N), \forall d \in \left( \frac{N}{N-2s}, 2^*_{\alpha,s} \right) \), we get \( |\Psi_{\varepsilon,b}|^2 \in L^t(\mathbb{R}^N) \). Then, using Hölder’s inequality and change of variable, we obtain
\[
\int_{\mathbb{R}^N} a(x) |\Psi_{e,b}|^2 \, dx \leq |a|_q \left( \int_{\mathbb{R}^N} \left| \frac{B_{\alpha,s} \varepsilon^{N-2s} x}{|x|^2 + |b|^2} \right|^{2t} \, dx \right)^{\frac{1}{t}}
\]

\[
= |a|_q \left( \int_{\mathbb{R}^N} \left| \frac{B_{\alpha,s} \varepsilon^{N-2s} x}{|x|^2 + |z|^2} \right|^{2t} \, dz \right)^{\frac{1}{t}}
\]

\[
= |a|_q \left( \int_{\mathbb{R}^N} |\Psi_{e,0}|^{2t} \, dx \right)^{\frac{1}{t}} = |a|_q |\Psi_{e,0}|^{2t}, \quad \forall b \in \mathbb{R}^N.
\]

From Lemma 4.1-(iii), given \( \tau > 0 \), there exists \( \varepsilon = \varepsilon(\tau) > 0 \) such that

\[
\sup_{b \in \mathbb{R}^N} f(\Psi_{e,b}) \leq S_{h,l} + \frac{\tau}{2} < S_{h,l} + \tau, \quad \forall \varepsilon \in (0, \varepsilon].
\]

Now, suppose that \( q \in \left[p_1, \frac{2N-\alpha}{4s-\alpha}\right) \) with \( t \in (0, \infty) \) and \( \frac{1}{q} + \frac{1}{t} = 1 \). Notice that \( 2t - 2_{a,s}^* > 0 \) and for \( \varepsilon > 1 \),

\[
|\Psi_{e,b}| \in L^\infty(\mathbb{R}^N)
\]

and \( |\Psi_{e,b}|^{2_{a,s}^*} \in L^1(\mathbb{R}^N) \). Then \( |\Psi_{e,b}|^2 \in L^t(\mathbb{R}^N) \). Applying again Hölder’s inequality with \( q \) and \( t \), we get

\[
\int_{\mathbb{R}^N} a(x) |\Psi_{e,b}|^2 \, dx \leq |a|_q \left( \int_{\mathbb{R}^N} |\Psi_{e,0}|^{2t} \, dx \right)^{\frac{1}{t}}
\]

\[
= |a|_q \left( \int_{\mathbb{R}^N} |\Psi_{e,0}|^{2t - 2_{a,s}^*} |\Psi_{e,0}|^{2_{a,s}^*} \, dx \right)^{\frac{1}{t}}
\]

\[
\leq |a|_q |\Psi_{e,0}|^{(2t - 2_{a,s}^*)/t} \left( \int_{\mathbb{R}^N} |\Psi_{e,0}|^{2_{a,s}^*} \, dx \right)^{\frac{1}{t}}
\]

\[
\leq C |a|_q \varepsilon^{\frac{2t-N - 2_{a,s}^*}{2t}}, \quad \forall b \in \mathbb{R}^N.
\]

Then, given \( \tau > 0 \), there is \( \tilde{\varepsilon} = \tilde{\varepsilon}(\tau) > 0 \) such that

\[
\varepsilon^{\frac{2t-N - 2_{a,s}^*}{2t}} < \frac{\tau}{2C |a|_q}, \quad \forall \varepsilon \in [\tilde{\varepsilon}, \infty),
\]

which implies
\[ f(\Psi_{\varepsilon,b}) = \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}|^2 + a(x) |\Psi_{\varepsilon,b}|^2 \right) dx \]
\[ \leq S_{h,l} + \int_{\mathbb{R}^N} a(x) |\Psi_{\varepsilon,b}|^2 dx \]
\[ \leq S_{h,l} + \sup_{b \in \mathbb{R}^N} \int_{\mathbb{R}^N} a(x) |\Psi_{\varepsilon,b}|^2 dx \]
\[ \leq S_{h,l} + \frac{\tau}{2} \leq S_{h,l} + \tau, \quad \forall b \in \mathbb{R}^N \text{ and } \forall \varepsilon \in [\varepsilon, \infty). \]

The proof is now complete. \( \square \)

**Lemma 4.4.** Suppose that \( |a|_{L^N(2\varepsilon) \mathbb{R}^N} < \left( 2^{\frac{4s-a}{2N-a}} - 1 \right) S \frac{(2s-N)(N-a)(1+2\varepsilon)+2(2N-a)2s}{2s(N-a)+2s} \). Then

\[ \sup_{\varepsilon > 0, b \in \mathbb{R}^N} \ f(\Psi_{\varepsilon,b}) < 2^{\frac{4s-a}{2N-a}} S_{h,l}. \]

**Proof.** From (2.8)-(2.10), (4.1)-(4.2) and (2.5), a direct calculation yields that

\[ |\Psi_{\varepsilon,b}|^2 \leq \left( S_{h,l} \frac{N-2s}{N+2s-a} S \frac{(N-a)(2s-N)}{4(2s-a+2s)} D_{N,a} \right)^2 \left[ \int_{\mathbb{R}^N} |U_{\varepsilon,b}(x)|^2 \right] \frac{2s}{s+N} \]
\[ = S_{h,l} \left( \frac{N-2s}{N+2s-a} S \frac{(2s-N)(N-a)}{4(2s-a+2s)} D_{N,a} \right)^2 S_{h,l} \frac{2s}{s+N} \]
\[ = S_{h,l} S \frac{(N-2s)(N-a)(1+2\varepsilon)+2(2N-a)2s}{2s(N-a)+2s} . \]

Hence, by Hölder’s inequality with \( \frac{N}{2s} \) and \( \frac{N}{N-2s} \), we obtain

\[ \sup_{\delta > 0, b \in \mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + |a|_{\mathbb{R}^N} |\Psi_{\varepsilon,b}|^2 < S_{h,l} + \left( 2^{\frac{4s-a}{2N-a}} - 1 \right) S_{h,l} = 2^{\frac{4s-a}{2N-a}} S_{h,l}. \]

This completes the proof. \( \square \)

In what follows, we consider the function

\[ \xi(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ 1, & \text{if } |x| \geq 1 \end{cases} \]

and define \( \alpha : D^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}^{N+1} \) by

\[ \alpha(u) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \left( \frac{x}{|x|} , \xi(x) \right) \left( \frac{\beta(u), \gamma(u)}{|(\Delta)^{\frac{s}{2}} u|^2} \right) dx = (\beta(u), \gamma(u)). \]
where
\[
\beta(u) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \frac{x}{|x|} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx
\]
and
\[
\gamma(u) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \xi(x) |(-\Delta)^{\frac{s}{2}} u|^2 \, dx.
\]

**Lemma 4.5.** If $|b| \geq \frac{1}{2}$, then
\[
\beta(\Psi_{1\varepsilon,b}) = \frac{b}{|b|} + o_\varepsilon(1) \quad \text{when} \quad \varepsilon \to 0.
\]

**Proof.** Given $\varepsilon > 0$, from Lemma 4.2, there is $\tilde{\varepsilon} > 0$ such that
\[
\frac{1}{S_{h,l}} \int_{\mathbb{R}^N \setminus B_r(b)} |(-\Delta)^{\frac{s}{2}} \psi_{\varepsilon,b}|^2 \, dx = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N \setminus B_r(0)} |(-\Delta)^{\frac{s}{2}} \psi_{\varepsilon,0}|^2 \, dx
\]
\[
\leq C \tilde{C}^2 \int_{\mathbb{R}^N \setminus B_r(0)} \frac{\varepsilon^{N-2s} |x|^2}{[\varepsilon^2 + |x|^2]^{N-2s+2}}
\]
\[
= C \tilde{C}^2 \varepsilon^{2s-2} \int_{\mathbb{R}^N \setminus B_{\varepsilon r,0}(0)} \frac{|z|^2}{[1 + |z|^2]^{N-2s+2}} \, dz
\]
\[
\leq C_1 \varepsilon^{N-2s} r^{-N-2+4s} < \tau, \quad \forall \varepsilon \in (0, \tilde{\varepsilon}).
\]

Thus,
\[
\left| \beta(\Psi_{\varepsilon,b}) - \frac{1}{S_{h,l}} \int_{B_r(b)} \frac{x}{|x|} |(-\Delta)^{\frac{s}{2}} \psi_{\varepsilon,b}|^2 \, dx \right| \leq \frac{1}{S_{h,l}} \int_{\mathbb{R}^N \setminus B_r(b)} |(-\Delta)^{\frac{s}{2}} \psi_{\varepsilon,b}|^2 \, dx
\]
\[
< \tau, \quad \forall \varepsilon \in (0, \tilde{\varepsilon}).
\]

Note that if $\varepsilon$ is small enough, for $|b| \geq \frac{1}{2}$ and $x \in B_r(b)$,
\[
\left| \frac{x}{|x|} - \frac{b}{|b|} \right| < 2\tau.
\]

Then, by (4.11), (4.12), we have
From (4.12) and (4.13), we get

$$\left| \beta(\psi_{1,\varepsilon}) - \frac{b}{|b|} \right| < 4\tau,$$

which completes the proof. □

**Lemma 4.6.** Suppose that $a \in L^q(\mathbb{R}^N)$, $\forall q \in [p_1, p_2]$, where $1 < p_1 < \frac{2N-a}{4s-\alpha} < p_2 < \frac{N}{4s-N}$ if $2s < N < 4s$. Then for every $\varepsilon > 0$, we have

$$\lim_{|b| \to \infty} f(\psi_{1,\varepsilon}) = S_{h,l}.$$

**Proof.** Since

$$f(\psi_{1,\varepsilon}) = \int_{\mathbb{R}^N} \left( |(-\Delta)^{\frac{s}{2}} \psi_{1,\varepsilon}|^2 + a(x)|\psi_{1,\varepsilon}|^2 \right) dx = S_{h,l} + \int_{\mathbb{R}^N} a(x)|\psi_{1,\varepsilon}|^2 dx,$$

it is sufficient to prove that

$$\lim_{|b| \to \infty} \int_{\mathbb{R}^N} a(x)|\psi_{1,\varepsilon}|^2 dx \to 0, \quad \forall \varepsilon > 0. \quad (4.14)$$

Notice that for given $\tau > 0$, there exists $r_0 > 0$ such that

$$\left( \int_{\mathbb{R}^N \setminus B_{r_0}(0)} |a(x)|^{\frac{2N-a}{4s-\alpha}} \frac{4s-a}{2N-a} \right)^{\frac{4s-a}{2N-a}} < \tau, \quad \forall \rho > r_0,$$

and
Let $r_0 < 2 \rho < |b|$, and then $B_\rho(0) \cap B_\rho(b) = \emptyset$. By (4.6) and Hölder’s inequality we have

\[
\int a(x) |\Psi_{\epsilon,b}|^2 \, dx \\
\leq \left( \int_{\mathbb{R}^N \setminus (B_\rho(0) \cup B_\rho(b))} |a(x)|^{\frac{2N-a}{4s-4a}} \, dx \right)^{\frac{4s-4a}{2N-a}} \left( \int_{\mathbb{R}^N \setminus (B_\rho(0) \cup B_\rho(b))} |\Psi_{\epsilon,b}|^{2^*_{\alpha,s}} \, dx \right)^{\frac{2}{2^*_{\alpha,s}}} \\
+ \left( \int_{B_\rho(0)} |a(x)|^{\frac{2N-a}{4s-4a}} \, dx \right)^{\frac{4s-4a}{2N-a}} \left( \int_{B_\rho(0)} |\Psi_{\epsilon,b}|^{2^*_{\alpha,s}} \, dx \right)^{\frac{2}{2^*_{\alpha,s}}} \\
+ \left( \int_{B_\rho(b)} |a(x)|^{\frac{2N-a}{4s-4a}} \, dx \right)^{\frac{4s-4a}{2N-a}} \left( \int_{B_\rho(b)} |\Psi_{\epsilon,b}|^{2^*_{\alpha,s}} \, dx \right)^{\frac{2}{2^*_{\alpha,s}}} \\
< \tau \tau^2 + \tau^2 |a|^{\frac{2N-a}{4s-4a}} + \tau C \epsilon^{\frac{a}{2}}
\]

which completes the proof. \(\square\)

Now we define the set

\[
\mathcal{Z} := \left\{ u \in \mathcal{M} : \alpha(u) = \left( 0, \frac{1}{2} \right) \right\}.
\]

**Lemma 4.7.** The set \(\mathcal{Z}\) is not empty, that is, \(\mathcal{Z} \neq \emptyset\).
Proof. Since the function $\Psi_{\epsilon,0}$ is symmetric about the origin, we have $\beta(\Psi_{\epsilon,0}) = 0$. From Lemma 4.2 we see that

$$\gamma(\Psi_{\epsilon,0}) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N \setminus B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\epsilon,0}|^2 dx \to 0 \quad \text{as} \quad \epsilon \to 0. \quad (4.16)$$

Note that,

$$\gamma(\Psi_{\epsilon,0}) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \xi(x) |(-\Delta)^{\frac{s}{2}} \Psi_{\epsilon,0}|^2 dx$$

$$= \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_{\epsilon,0}|^2 dx + \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} (\xi(x) - 1) |(-\Delta)^{\frac{s}{2}} \Psi_{\epsilon,0}|^2 dx$$

$$= 1 - \frac{1}{S_{h,l}} \int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\epsilon,0}|^2 dx. \quad (4.17)$$

From (4.5) and Proposition 2.2 [39], we have

$$\int_{B_1(0)} |(-\Delta)^{\frac{s}{2}} \Psi_{\epsilon,0}|^2 dx \leq C \int_{B_1(0)} |\nabla \Psi_{\epsilon,0}|^2 dx$$

$$\leq C \int_{B_1(0)} \frac{\epsilon^{N-2s} |x|^2}{[\epsilon^2 + |x|^2]^{N-2s+2}}$$

$$= C \epsilon^{2s-2} \int_{B_1(0)} \frac{|z|^2}{[1 + |z|^2]^{N-2s+2}} dz$$

$$\leq C_1 \epsilon^{2s-2} \to 0 \quad \text{as} \quad \epsilon \to \infty. \quad (4.18)$$

Combining (4.17) and (4.18) we get

$$\gamma(\Psi_{\epsilon,0}) \to 1 \quad \text{as} \quad \epsilon \to \infty. \quad (4.19)$$

By (4.16) and (4.19) we see that there is a $\epsilon_1 > 0$ such that $\Psi_{\epsilon_1,0} \in \mathcal{Z}$. \qed

Lemma 4.8. The number $c_0 = \inf_{u \in \mathcal{Z}} f(u)$ satisfies the inequality $c_0 > S_{h,l}$.

Proof. Since $\mathcal{Z} \subset \mathcal{M}$, we have

$$S_{h,l} \leq c_0.$$  

Suppose, by contradiction, that $S_{h,l} = c_0$. By Ekeland variational principle [51], there exists $(u_n) \subset D^{s,2}(\mathbb{R}^N)$ such that
\[
\int_{\mathbb{R}^N} (I_{\alpha} * |u_n|^{2s_{\alpha,s}}) |u_n|^{2s_{\alpha,s}} \, dx = 1, \quad \alpha(u_n) \to \left( 0, \frac{1}{2} \right)
\]  
(4.20)

and

\[
f(u_n) \to S_{h,l}, \quad f'_{|_{\mathcal{M}}} (u_n) \to 0.
\]  
(4.21)

Thus, \((u_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\) and, up to a subsequence, \(u_n \rightarrow u_0\) in \(D^{s,2}(\mathbb{R}^N)\).

If \(v_n = S_{h,l}^{(N-2s)/(2N+4s-2\alpha)} u_n\) and \(v_0 = S_{h,l}^{(N-2s)/(2N+4s-2\alpha)} u_0\), we have that \(v_n \rightarrow v_0\) in \(D^{s,2}(\mathbb{R}^N)\). Furthermore, from (4.21) and Lemma 3.5 we obtain

\[
I(v_n) \rightarrow \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha} \quad \text{and} \quad I'(v_n) \rightarrow 0.
\]

We are going to show that \(v_0 \equiv 0\). First of all, note that

\[
u_n \not\rightarrow u_0 \quad \text{in} \quad D^{s,2}(\mathbb{R}^N),
\]  
(4.22)

because otherwise, \(u_0 \neq 0\) and

\[
S_{h,l} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 \, dx
\]

\[
< \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 \, dx + \int_{\mathbb{R}^N} a(x) |u_0|^2 \, dx = S_{h,l},
\]

which is a contradiction. Thus, \(v_n \not\rightarrow v_0\) in \(D^{s,2}(\mathbb{R}^N)\) and since \((v_n)\) is a \((PS)_c\) sequence for \(I\), by Theorem 3.2 we have

\[
I(v_n) \rightarrow I(v_0) + \sum_{j=1}^{k} I_{z_0^j} = \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha}.
\]

Using \(I_{z_0^1} = 0\), we get

\[
I(v_0) = 0, \quad k = 1, \quad z_0^1 > 0, \quad \text{and} \quad I_{z_0^1} = \frac{N + 2s - \alpha}{4N - 2\alpha} S_{h,l}^{2N-\alpha}.
\]  
(4.23)

By virtue of \(v_0\) being weak solution of \((P)\), we have

\[
I(v_0) = \frac{N + 2s - \alpha}{4N - 2\alpha} \int_{\mathbb{R}^N} (I_{\alpha} * |v_0|^{2s_{\alpha,s}}) |v_0|^{2s_{\alpha,s}} \, dx = 0,
\]

which implies that \(v_0 \equiv 0\). Then, \((v_n)\) is a \((PS)_c\) sequence for \(I\) such that \(v_n \not\rightarrow 0\), \(v_n \not\rightarrow 0\) and \(\int_{\mathbb{R}^N} a(x) |v_n|^2 \, dx = o_n(1)\). Therefore,
\[
\frac{N + 2s - \alpha}{4N - 2\alpha} S^{\frac{N-2s}{4N-2\alpha}}_{h,l} + o_n(1) = I(v_n) = I_{\infty}(v_n) + \int_{\mathbb{R}^N} a(x)|v_n|dx = I_{\infty}(v_n) + o_n(1) \tag{4.24}
\]

and
\[
\|I_{\infty}'(v_n)\|_{D^{-\frac{s}{2}}(\mathbb{R}^N)} \leq \|I'(v_n)\|_{D^{-\frac{s}{2}}(\mathbb{R}^N)} + o_n(1). \tag{4.25}
\]

From (4.24) and (4.25) we conclude that \((v_n)\) is a \((PS)_c\) sequence for \(I_{\infty}\) and by Lemma 3.1, there are sequences \((R_n) \subset \mathbb{R}, (x_n) \subset \mathbb{R}^N, z_0^1\) nontrivial solution of \((P_{\infty})\) and \((w_n)\) a \((PS)_c\) sequence for \(I_{\infty}\) such that
\[
v_n(x) = w_n(x) + R_n^{(N-2s)/2} z_0^1(R_n(x - x_n)) + o_n(1). \tag{4.26}
\]

Setting
\[
z_n(x) = R_n^{(N-2s)/2} z_0^1(R_n(x - x_n)),
\]
and making change of variable, we have
\[
I_{\infty}'(z_n)\varphi = I_{\infty}'(z_0^1)\varphi_n = 0, \quad \forall \varphi \in D^{s,2}(\mathbb{R}^N), \quad \forall n \in \mathbb{N},
\]
i.e., \(z_n\) is a solution of \((P_{\infty})\), for all \(n \in \mathbb{N}\). From \(z_0^1\) solving (2.7) and (2.9) we know that there exist some \(\varepsilon_n > 0, b_n \in \mathbb{R}^N\) such that
\[
z_n(x) = b_{\alpha,s} \left( \frac{\varepsilon_n}{\varepsilon_n^2 + |x - b_n|^2} \right)^{\frac{N-2s}{2}}, \quad \forall x \in \mathbb{R}^N.
\]

By (4.26), (4.1), we get
\[
u_n(x) = \tilde{w}_n(x) + \Psi_{\varepsilon_n, b_n}(x) + o_n(1),
\]
where
\[
\tilde{w}_n(x) = S_{h,l}^{\frac{N-2s}{4N-2\alpha}} w_n(x), \quad \Psi_{\varepsilon_n, b_n}(x) = S_{h,l}^{\frac{N-2s}{4N-2\alpha}} z_n(x).
\]

By (4.23) we derive that \(w_n \to 0\), which implies that \(\tilde{w}_n \to 0\) in \(D^{s,2}(\mathbb{R}^N)\). Therefore, from (4.20), we have
\[
\left(0, \frac{1}{2}\right) + o_n(1) = \alpha(u_n) = \alpha(\tilde{w}_n + \Psi_{\varepsilon_n, b_n} + o_n(1)) = \alpha(\Psi_{\varepsilon_n, b_n})
\]
which implies that
\[
(i) \quad \beta(\Psi_{\varepsilon_n, b_n}) \to 0
\]
and
\[(ii) \ \gamma(\Phi_{\epsilon_n,b_n}) \to \frac{1}{2}.\]

Passing to a subsequence, one of the following cases must occur.

(a) \(\epsilon_n \to +\infty\) when \(n \to +\infty\);
(b) \(\epsilon_n \to \bar{\epsilon} \neq 0\) when \(n \to +\infty\);
(c) \(\epsilon_n \to 0\) and \(b_n \to \bar{b}\) when \(n \to +\infty\) with \(|\bar{b}| < \frac{1}{2}\);
(d) \(\epsilon_n \to 0\) when \(n \to +\infty\) and \(|b_n| \geq \frac{1}{2}\) for \(n\) sufficiently large.

Suppose that (a) is true. Then
\[
\gamma(\Phi_{\epsilon_n,b_n}) = 1 - \frac{1}{S_{h,l}} \int_{B(0)} |(-\Delta)^{\frac{s}{2}} \Phi_{\epsilon_n,b_n}|^2 dx
\]
with Lemma 4.1 gives
\[
|\gamma(\Phi_{\epsilon_n,b_n}) - 1| = \frac{1}{S_{h,l}} \int_{B(0)} |(-\Delta)^{\frac{s}{2}} \Phi_{\epsilon_n,b_n}|^2 dx = o_n(1),
\]
which contradicts (ii).

Suppose that (b) is true. In this case we may suppose that \(|b_n| \to +\infty\) because if \(b_n \to \bar{b}\), we can prove that
\[
\Phi_{\epsilon_n,b_n} \to \Phi_{\bar{\epsilon},\bar{b}} \quad \text{in} \quad D^{s,2}(\mathbb{R}^N).
\]
Since \(\tilde{w}_n \to 0\) in \(D^{s,2}(\mathbb{R}^N)\) and \(u_n = \tilde{w}_n + \Phi_{\epsilon_n,b_n} + o_n(1)\), we see that \((u_n)\) converges in \(D^{s,2}(\mathbb{R}^N)\) but this is a contradiction with (4.22). Hence,
\[
\gamma(\Phi_{\epsilon_n,b_n}) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \xi(x)|(-\Delta)^{\frac{s}{2}} \Phi_{\epsilon_n,b_n}|^2 dx
\]
\[
= \frac{1}{S_{h,l}} \int_{\mathbb{R}^N \setminus B(0)} |(-\Delta)^{\frac{s}{2}} \Phi_{\epsilon_n,b_n}|^2 dx
\]
\[
= 1 - \frac{1}{S_{h,l}} \int_{B(0)} |(-\Delta)^{\frac{s}{2}} \Phi_{\epsilon_n,0}|^2 dx. \quad (4.27)
\]
Applying Lebesgue’s theorem we can show that
\[
\int_{B_1(-b_n)} |(-\Delta)^{\frac{s}{2}} \Phi_{\epsilon_n,0}|^2 dx \to 0 \quad \text{as} \quad n \to +\infty
\]
and from (4.27) we obtain
\( \gamma(\Psi_{\varepsilon_n, b_n}) \to 1 \) when \( n \to +\infty \),

which contradicts to \((ii)\).

Suppose that \((c)\) is true. Note that

\[
\gamma(\Psi_{\varepsilon_n, b_n}) = \frac{1}{\mathcal{S}_{h,l}} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon_n, b_n}|^2 \, dx
\]

(4.28)

Therefore, using again the Lebesgue theorem, we find

\[
\lim_{n \to +\infty} \int_{B_1(-b_n)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon_n, 0}|^2 \, dx = \mathcal{S}_{h,l}.
\]

From (4.28) we get

\[
\gamma(\Psi_{\varepsilon_n, b_n}) \to 0,
\]

which is a contradiction with \((ii)\).

Suppose that \((d)\) is true. Since \( |b_n| \geq \frac{1}{2} \) for \( n \) large, we have that \( b_n \not\to 0 \) in \( \mathbb{R}^N \). From Lemma 4.5 we have

\[
\beta(\Psi_{\varepsilon_n, b_n}) = \frac{b_n}{|b_n|} + o_n(1).
\]

Thus,

\[
\beta(\Psi_{\varepsilon_n, b_n}) \not\to 0
\]

which is a contradiction with \((i)\). So, \( \mathcal{S}_{h,l} < c_0 \) and the proof is completed. \( \Box \)

**Lemma 4.9.** There is \( \varepsilon_1 \in (0, 1/2) \) such that

(a) \( f(\Psi_{\varepsilon_1, b}) < \frac{\mathcal{S}_{h,l} + c_0}{2}, \forall b \in \mathbb{R}^N \); 

(b) \( \gamma(\Psi_{\varepsilon_1, b}) < \frac{1}{2}, \forall b \in \mathbb{R}^N \) such that \( |b| < \frac{1}{2} \); 

(c) \( \left| \beta(\Psi_{\varepsilon_1, b}) - \frac{b}{|b|} \right| < \frac{1}{4}, \forall b \in \mathbb{R}^N \) such that \( |b| \geq \frac{1}{2} \).

**Proof.** Using Lemma 4.3, with \( \tau = \frac{c_0 - \mathcal{S}_{h,l}}{2} > 0 \) and \( \varepsilon_1 < \min\{\varepsilon, 1/2\} \), we conclude that

\[
f(\Psi_{\varepsilon, b}) \leq \sup_{b \in \mathbb{R}^N} f(\Psi_{\varepsilon, b}) < \mathcal{S}_{h,l} + \frac{c_0 - \mathcal{S}_{h,l}}{2} = \mathcal{S}_{h,l} + \frac{c_0}{2}, \forall b \in \mathbb{R}^N,
\]

(4.29)
showing that (a) holds true. Now, be the definition of $\xi$,

$$\gamma(\Psi_{\varepsilon,b}) = 1 - \frac{1}{S_{h,l}} \int_{B_1(-b)} |(-\Delta)\hat{\Psi}_{\varepsilon,0}|^2 \, dx.$$ 

By Lebesgue’s theorem,

$$\lim_{\varepsilon \to 0} \int_{B_1(-b)} |(-\Delta)\hat{\Psi}_{\varepsilon,0}|^2 \, dx = S_{h,l}$$

proving (b).

Notice that from Lemma 4.5, we derive that

$$\beta(\Psi_{\varepsilon,b}) = \frac{b}{|b|} + o_\varepsilon(1) \quad \text{when} \quad \varepsilon \to 0, \quad \forall b \in \mathbb{R}^N; \quad |b| \geq \frac{1}{2}$$

and the proof is over. □

**Lemma 4.10.** There is $\varepsilon_2 > 1$ such that

(a) $f(\Psi_{\varepsilon_2,y}) < \frac{S_{h,l} + c_0}{2}$, \quad $\forall b \in \mathbb{R}^N$;

(b) $\gamma(\Psi_{\varepsilon_2,b}) > \frac{1}{2}$, \quad $\forall b \in \mathbb{R}^N$.

**Proof.** By Lemma 4.3, we can take $\tau = \frac{c_0 - S_{h,l}}{2} > 0$ and $\varepsilon_2 > \max\{\varepsilon, 1\}$ such that

$$f(\Psi_{\varepsilon,b}) \leq \sup_{b \in \mathbb{R}^N} f(\Psi_{\varepsilon,b}) < S_{h,l} + \frac{c_0 - S_{h,l}}{2} = \frac{S_{h,l} + c_0}{2}, \quad \forall b \in \mathbb{R}^N. \quad (4.30)$$

Moreover, by the definition of $\xi$ and Lemma 4.1, we see that

$$\gamma(\Psi_{\varepsilon,b}) \to 1 \quad \text{when} \quad \varepsilon \to +\infty$$

and the proof is over. □

**Lemma 4.11.** There exists $R > 0$ such that

(a) $f(\Psi_{\varepsilon,b}) < \frac{S_{h,l} + c_0}{2}$, \quad $\forall b; \quad |b| \geq R$ and $\varepsilon \in [\varepsilon_1, \varepsilon_2]$;

(b) $(\beta(\Psi_{\varepsilon,b})|b|_{\mathbb{R}^N} > 0$, \quad $\forall b; \quad |b| \geq R$ and $\varepsilon \in [\varepsilon_1, \varepsilon_2]$.

**Proof.** Since $\varepsilon$ varies in the compact set $[\varepsilon_1, \varepsilon_2]$ by Lemma 4.6 we can find an $R_1 > 0$, big enough, by choosing $\tau = \frac{c_0 - S_{h,l}}{2} > 0$ so that

$$f(\Psi_{\varepsilon,b}) < S_{h,l} + \tau = \frac{S_{h,l} + c_0}{2}, \quad \forall b; \quad |b| \geq R_1 \text{ and } \varepsilon \in [\varepsilon_1, \varepsilon_2],$$

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and item \((a)\) follows. Now we prove item \((b)\). Note that, for each \(y \in \mathbb{R}^N\) we consider the sets \((\mathbb{R}^N)^+_b = \{ x \in \mathbb{R}^N : \langle x | b \rangle_{\mathbb{R}^N} > 0 \}, (\mathbb{R}^N)^-_b = \mathbb{R}^N \setminus (\mathbb{R}^N)^+_b\). Since \(\varepsilon\) varies in the compact set \([\varepsilon_1, \varepsilon_2]\), we can prove there is \(R_2 > 0\) big enough and \(r \in (0, 1/4)\) such that the following things are true if \(|b| \geq R_2\) and \(|b - \tilde{b}| = 1/2\),

\[
B_r(\tilde{b}) = \{ x \in \mathbb{R}^N : |x - \tilde{b}| < r \} \subset (\mathbb{R}^N)^+_b.
\]

Note that for each \(x \in B_r(\tilde{b})\), by means of equality \((1.5)\) we have

\[
|(-\Delta)^{\frac{1}{2}} \Psi_{\varepsilon, b}(x)|^2 = \int_{\mathbb{R}^N} \left| \tilde{C}_\varepsilon \frac{N-2s}{2} \frac{\beta(\Psi_{1\varepsilon,b})}{\Delta_1} + \tilde{C}_\varepsilon \frac{N-2s}{2} \frac{\beta(\Psi_{1\varepsilon,b})}{\Delta_1} \right| \frac{1}{|x - z|^{N+2s}} \, dz
\]

\[
\geq \int_{B_{r/2}(\tilde{b}) \setminus B_{r/4}(\tilde{b})} \left| \tilde{C}_\varepsilon \frac{N-2s}{2} \frac{\beta(\Psi_{1\varepsilon,b})}{\Delta_1} + \tilde{C}_\varepsilon \frac{N-2s}{2} \frac{\beta(\Psi_{1\varepsilon,b})}{\Delta_1} \right| \frac{1}{|x - z|^{N+2s}} \, dz
\]

\[
\geq \frac{2\varepsilon_1^{N-2s}}{(7/8)^{N+2s}} \int_{B_{r/2}(\tilde{b}) \setminus B_{r/4}(\tilde{b})} \left| \tilde{C}_\varepsilon \frac{N-2s}{2} \frac{\beta(\Psi_{1\varepsilon,b})}{\Delta_1} + \tilde{C}_\varepsilon \frac{N-2s}{2} \frac{\beta(\Psi_{1\varepsilon,b})}{\Delta_1} \right| \frac{1}{|x - z|^{N+2s}} \, dz
\]

\[
= H_1 > 0.
\]

Therefore,

\[
\langle \beta(\Psi_{\varepsilon,b}) | b \rangle_{\mathbb{R}^N} \geq \frac{1}{S_{h,l}} \int_{B_r(\tilde{b})} \frac{\langle x | b \rangle_{\mathbb{R}^N}}{|x|} H_1 \, dx + \frac{1}{S_{h,l}} \int_{(\mathbb{R}^N)^+_b} \frac{\langle x | b \rangle_{\mathbb{R}^N}}{|x|} |(-\Delta)^{\frac{1}{2}} \Psi_{\varepsilon,b}(x)|^2 \, dx
\]

\[
= \frac{|b|}{S_{h,l}} \int_{B_r(\tilde{b})} \frac{\langle x | b \rangle_{\mathbb{R}^N}}{|x||b|} H_1 \, dx - \frac{|b|}{S_{h,l}} \int_{(\mathbb{R}^N)^+_b} \frac{H_1}{|x|} \, dx = H_2 > 0.
\]

Moreover, we have

\[
\frac{1}{S_{h,l}} \int_{B_r(\tilde{b})} \frac{\langle x | b \rangle_{\mathbb{R}^N}}{|x||b|} H_1 \, dx \geq \frac{C}{S_{h,l}} \int_{B_r(\tilde{b})} \frac{H_1}{|x|} \, dx = H_2 > 0.
\]

From Proposition 2.2 [39], (4.5) and (4.8), we can derive that
\[
\int_{(\mathbb{R}^N)_b^c} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}(x)|^2 dx \leq \int_{B_{R_2}^c(b)} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}(x)|^2 dx \leq C \int_{B_{R_2}^c(0)} |\nabla \Psi_{\varepsilon,b}|^2 dx
\]
(4.34)

It is possible to choose \( R_2 > 0 \) large, such that for all \( b \in \mathbb{R}^N \) with \( |b| > R_2 \), we have
\[
\frac{1}{S_{h,l}} \int_{(\mathbb{R}^N)_b^c} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}(x)|^2 dx < \frac{H_2}{2}.
\]
(4.35)

From (4.32)-(4.35), we get
\[
(\beta(\Psi_{\varepsilon,b})|b|)_{\mathbb{R}^N} \geq |b| \left\{ \frac{1}{S_{h,l}} \int_{B_{\varepsilon}(0)} \frac{|x||b|}{|x|} H_1 dx - \frac{1}{S_{h,l}} \int_{(\mathbb{R}^N)_b^c} |(-\Delta)^{\frac{s}{2}} \Psi_{\varepsilon,b}(x)|^2 dx \right\}
\]
(4.36)

for all \( b \in \mathbb{R}^N \) with \( |b| > R = \max\{R_1, R_2\} > 0 \) and for all \( \varepsilon \in [\varepsilon_1, \varepsilon_2] \), and the proof is over. \( \Box \)

5. Proof of the main theorem

In this section we prove Theorem 1.1. For this aim, we first fix some notations and give some more technical lemmas. Let us consider now the sets
\[
V = \{ (b, \varepsilon) \in \mathbb{R}^N \times (0, \infty) : |b| < R \text{ and } \varepsilon \in (\varepsilon_1, \varepsilon_2) \},
\]
where \( \varepsilon_1, \varepsilon_2 \) and \( R \) are given in Lemmas 4.9, 4.10 and 4.11, respectively. Moreover, we define the continuous function \( Q : \mathbb{R}^N \times (0, \infty) \to D^{1,2}(\mathbb{R}^N) \) given by
\[
Q(b, \varepsilon) = \Psi_{\varepsilon,b}.
\]
Under the above notations, we also denote the sets
\[
\Theta = \{ Q(b, \varepsilon) : (b, \varepsilon) \in V \},
\]
\[
\mathcal{H} = \left\{ h \in C(\Sigma \cap \mathcal{M}, \Sigma \cap \mathcal{M}) : h(u) = u, \forall u \in (\Sigma \cap \mathcal{M}) : f(u) < \frac{S_{h,l} + c_0}{2} \right\},
\]
and
\[ \Gamma = \{ A \subseteq (\Sigma \cap M) : A = h(\Theta), h \in \mathcal{H} \}. \]

Clearly, \( \Theta \subseteq (\Sigma \cap M) \), \( \Theta = Q(V) \) is compact and \( \mathcal{H} \neq \emptyset \), because the identity function is in \( \mathcal{H} \).

**Lemma 5.1.** Let \( \mathcal{F} : V \rightarrow \mathbb{R}^{N+1} \) be a function given by

\[ \mathcal{F} = (\alpha \circ Q)(b, \varepsilon) = \frac{1}{S_{h,l}} \int_{\mathbb{R}^N} \left( \frac{x}{|x|}, \xi(x) \right) \left| (-\Delta)_{\varepsilon, b}^2 \Psi_{\varepsilon, b} \right|^2 dx. \]

Then

\[ \text{deg}(\mathcal{F}, V, (0, 1/2)) = 1. \]

**Proof.** Let \( G : [0, 1] \times V \rightarrow \mathbb{R}^{N+1} \) be the homotopy given by

\[ G(t, (b, \varepsilon)) = t\mathcal{F}(b, \varepsilon) + (1 - t)I_V(b, \varepsilon), \]

where \( I_V(b, \varepsilon) \) is the identity. We are going to show that \( (0, 1/2) \notin G([0, 1] \times (\partial V)) \), i.e.,

\[ t\beta(\Psi_{\varepsilon, b}) + (1 - t)\varepsilon \neq 0, \quad \forall t \in [0, 1] \quad \text{and} \quad \forall (b, \varepsilon) \in \partial V \]  

or

\[ t\gamma(\Psi_{\varepsilon, b}) + (1 - t)\varepsilon \neq \frac{1}{2}, \quad \forall t \in [0, 1] \quad \text{and} \quad \forall (b, \varepsilon) \in \partial V. \]

Notice that \( \partial V = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \), where

\[ \Gamma_1 = \{(b, \varepsilon_1) : |b| < 1/2\}, \]
\[ \Gamma_2 = \{(b, \varepsilon_1) : 1/2 \leq |b| \leq R\}, \]
\[ \Gamma_3 = \{(b, \varepsilon_2) : |b| \leq R\}, \]

and

\[ \Gamma_4 = \{(b, \varepsilon) : |b| = R, \quad \text{and} \quad \varepsilon \in [\varepsilon_1, \varepsilon_2]\}. \]

If \( (b, \varepsilon) \in \Gamma_1 \), then \( (b, \varepsilon) = (b, \varepsilon_1) \). From Lemma 4.9, \( \varepsilon_1 < \frac{1}{2} \) and \( \gamma(\Psi_{\varepsilon_1, b}) < \frac{1}{2} \). Hence,

\[ t\gamma(\Psi_{\varepsilon, b}) + (1 - t)\varepsilon = t\gamma(\Psi_{\varepsilon_1, b}) + (1 - t)\varepsilon_1 < \frac{t}{2} + \frac{1 - t}{2} = 1/2, \quad \forall t \in [0, 1] \quad \text{and} \quad \forall (b, \varepsilon) \in \Gamma_1 \]

showing that (5.2) is true and \((0, 1/2) \notin G([0, 1] \times \Gamma_1)\).

If \( (b, \varepsilon) \in \Gamma_2 \), then \( (b, \varepsilon) = (b, \varepsilon_1) \) and \( |b| \geq \frac{1}{2} \). Using Triangle inequality and by Lemma 4.9-(c) we derive
\[
|t\beta(\Psi_{\varepsilon,b}) - (1 - t)b| \geq \left| (1 - t)b + \frac{tb}{|b|} - \frac{tb}{|b|} - t\beta(\Psi_{\varepsilon_0,b}) \right|
\]
\[
= \left| \frac{(1 - t)|b| + t}{|b|} - t \left( \frac{b}{|b|} - \beta(\Psi_{\varepsilon_0,b}) \right) \right|
\]
\[
= |(1 - t)|b| + t| - t \left( \frac{b}{|b|} - \beta(\Psi_{\varepsilon_0,b}) \right) |
\]
\[
= \left( 1 - t \right)|b| + t - t \left( \frac{b}{|b|} - \beta(\Psi_{\varepsilon_0,b}) \right) |
\]
\[
\geq (1 - t)|b| + t - \frac{t}{4} \geq \frac{1}{2} + \frac{t}{4} > 0 \quad \forall t \in [0,1] \quad \text{and} \quad \forall (b, \varepsilon) \in \Gamma_2,
\]
showing that (5.1) is true and \((0, 1/2) \notin G([0, 1] \times \Gamma_2)\).

If \((\varepsilon, \varepsilon) \in \Gamma_3\), then \((b, \varepsilon) = (b, \varepsilon_2)\). By Lemma 4.10, \(\varepsilon_2 > \frac{1}{2}\) and \(\gamma(\Psi_{\varepsilon_2,b}) > \frac{1}{2}\), and so,
\[
t\gamma(\Psi_{\varepsilon,b}) + (1 - t)\varepsilon = t\gamma(\Psi_{\varepsilon_2,b}) + (1 - t)\varepsilon_2 > \frac{t}{2} + \frac{1 - t}{2} = \frac{1}{2}, \quad \forall t \in [0,1] \quad \text{and} \quad \forall (b, \varepsilon) \in \Gamma_3,
\]
showing that (5.2) is true and \((0, 1/2) \notin G([0, 1] \times \Gamma_3)\).

If \((b, \varepsilon) \in \Gamma_4\), we must have \(|b| = R\). Thus, by Lemma 4.11-(b), \((\beta(\Psi_{\varepsilon,b})\varepsilon)_{\mathbb{R}^N} > 0\). From this,
\[
(t\beta(\Psi_{\varepsilon,b}) + (1 - t)b)_{\mathbb{R}^N} = t(\beta(\Psi_{\varepsilon,b})b)_{\mathbb{R}^N} + (1 - t)(b)_{\mathbb{R}^N} > 0, \quad \forall t \in [0,1].
\]
Therefore (5.1) is true and \((0, 1/2) \notin G([0, 1] \times \Gamma_4)\).

The previous analysis ensures that \((0, 1/2) \notin G([0, 1] \times \partial V)\), then by properties of the topological degree
\[
\text{deg}(\mathcal{F}, V, (0, 1/2)) = \text{deg}(I_{\mathcal{F}}, V, (0, 1/2)).
\]
Since \((0, 1/2) \in V\), we deduce that
\[
\text{deg}(\mathcal{F}, V, (0, 1/2)) = \text{deg}(I_{\mathcal{F}}, V, (0, 1/2)) = 1. \quad \square
\]

**Lemma 5.2.** If \(A \in \Gamma\), then \(A \cap Z \neq \emptyset\).

**Proof.** It is sufficient to prove that for all \(h \in H\), there exists \((b_0, \varepsilon_0) \in \overline{V}\) such that
\[
(\alpha \circ h \circ Q)(b_0, \varepsilon_0) = \left( 0, \frac{1}{2} \right).
\]
Given \(h \in H\), let
\[
\mathcal{F}_h : \overline{V} \rightarrow \mathbb{R}^{N+1}
\]
be the continuous function given by
\[ F_h(b, \varepsilon) = (\alpha \circ h \circ Q)(b, \varepsilon). \]

We are going to prove that \( F_h = F \) in \( \partial V \). Note that

\[ \partial V = L_1 \cup L_2 \cup L_3, \tag{5.3} \]

where

\[ L_1 = \{(b, \varepsilon_1) : |b| \leq R\}, \quad L_2 = \{(b, \varepsilon_2) : |b| \leq R\} \]

and

\[ L_3 = \{(b, \varepsilon) : |b| = R \text{ and } \varepsilon \in [\varepsilon_1, \varepsilon_2]\}. \]

If \((b, \varepsilon) \in L_1\), then \((b, \varepsilon) = (b, \varepsilon_1)\) and by Lemma 4.9-(a), we get

\[ f(Q(b, \varepsilon)) = f(Q(b, \varepsilon_1)) = f(\Psi_{\varepsilon_1, b}) < \frac{S_{h,l} + c_0}{2}, \quad \forall (b, \varepsilon) \in L_1. \tag{5.4} \]

If \((b, \varepsilon) \in L_2\), then \((b, \varepsilon) = (b, \varepsilon_2)\) and by Lemma 4.10-(a), we get

\[ f(Q(b, \varepsilon)) = f(Q(b, \varepsilon_2)) = f(\Psi_{\varepsilon_2, b}) < \frac{S_{h,l} + c_0}{2}, \quad \forall (b, \varepsilon) \in L_2. \tag{5.5} \]

If \((b, \varepsilon) \in L_3\), then \(|b| = R\) and by Lemma 4.11-(a), we have

\[ f(Q(b, \varepsilon)) = f(\Psi_{\varepsilon, b}) < \frac{S_{h,l} + c_0}{2}, \quad \forall (b, \varepsilon) \in L_3. \tag{5.6} \]

By (5.4)-(5.6), we derive that

\[ f(Q(b, \varepsilon)) < \frac{S_{h,l} + c_0}{2}, \quad \forall (b, \varepsilon) \in \partial V. \]

Thus,

\[ F_h(b, \varepsilon) = (\alpha \circ h \circ Q)(b, \varepsilon) = (\alpha \circ h)Q(b, \varepsilon) = \alpha(h(Q(b, \varepsilon))) = (\alpha \circ Q)(b, \varepsilon) = F(b, \varepsilon), \quad \forall (b, \varepsilon) \in \partial V. \]

In view of \((0, 1/2) \notin F(\partial V)\), we have

\[ \deg(F, V, (0, 1/2)) = \deg(F_h, V, (0, 1/2)). \]

From Lemma 5.1, we get

\[ \deg(F_h, V, (0, 1/2)) = \deg(F, V, (0, 1/2)) = 1, \]

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and there exists \((b_0, \varepsilon_0) \in V\) such that
\[
F_h(b_0, \varepsilon_0) = (\alpha \circ h \circ Q)(b_0, \varepsilon_0) = \left(0, \frac{1}{2}\right)
\]
and the proof is over. \(\square\)

**Proof of Theorem 1.1.** Let us define the number
\[
c = \inf_{A \in \Gamma} \sup_{u \in A} f(u).
\]
Our first step is to prove that
\[
S_{h, l} < c < 2^{\frac{4l-\alpha}{2N}} S_{h, l}.
\] (5.7)
Note that, by Lemma 4.4,
\[
c = \inf_{A \in \Gamma} \sup_{u \in A} f(u) \leq \max_{u \in \Theta} f(u) \leq \sup_{y \in \mathbb{R}^N, \delta \in (0, \infty)} f(\Psi_1 \varepsilon, b) < 2^{\frac{4l-\alpha}{2N}} S_{h, l}.
\]
On the other hand, from Lemma 5.2 and Lemma 4.8 we get
\[
S_{h, l} < c_0 = \inf_{u \in \mathbb{Z}} f(u) \leq c = \inf_{A \in \Gamma} \sup_{u \in A} f(u) < 2^{\frac{4l-\alpha}{2N}} S_{h, l},
\] (5.8)
from where it follows (5.7).

Using the definition of \(c\), there exists \((u_n) \subset (\Sigma \cap \mathcal{M})\) such that
\[
f(u_n) \to c.
\] (5.9)
Suppose, by contradiction, that
\[
f'|_{\mathcal{M}}(u_n) \not\to 0.
\]
Then, there exists \((u_{n_j}) \subset (u_n)\) such that
\[
\|f'|_{\mathcal{M}}(u_{n_j})\|_* \geq l > 0, \ \forall j \in \mathbb{N}.
\]
By a Deformation Lemma \([51]\), there exists a continuous application \(\eta : [0, 1] \times (\Sigma \cap \mathcal{M}) \to (\Sigma \cap \mathcal{M}), \varepsilon_0 > 0\) such that

1. \(\eta(0, u) = u;\)
2. \(\eta(t, u) = u, \forall u \in f^{c-\varepsilon_0} \cup \{(\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0}\}, \forall t \in [0, 1];\)
3. \(\eta(1, f^{c+\varepsilon_0}/2) \subset f^{c-\varepsilon_0}/2.\)
From the definition of $c$, there exists $\hat{A} \in \Gamma$ such that
\[
c \leq \max_{u \in \hat{A}} f(u) < c + \frac{\varepsilon_0}{2},\]
where
\[
\hat{A} \subset f^{c + \varepsilon_0}. \tag{5.10}
\]
Since $\hat{A} \in \Gamma$, we have $\hat{A} \subset (\Sigma \cap M)$ and there exists $\tilde{h} \in H$ such that
\[
\tilde{h}(\Theta) = \hat{A}. \tag{5.11}
\]
From definition of $\eta$, we have
\[
\eta(1, \hat{A}) \subset (\Sigma \cap M). \tag{5.12}
\]
Let $h^*: (\Sigma \cap M) \to (\Sigma \cap M)$ be the function given by $h^*(u) = \eta(1, \tilde{h}(u))$ and note that $h^* \in C(\Sigma \cap M, \Sigma \cap M)$. We are going to show that
\[
f^{c + \varepsilon_0 \setminus f^{c - \varepsilon_0}} \subset f^{2(4s - a)(2N - a)S_{h,l}} \setminus f(S_{h,l} + c_0)/2. \tag{5.13}
\]
Indeed, given $u \in f^{c + \varepsilon_0 \setminus f^{c - \varepsilon_0}}$, we have
\[
c - \varepsilon_0 < f(u) \leq c + \varepsilon_0
\]
and by (5.8), for $\varepsilon_0$ sufficiently small, we get
\[
c - \varepsilon_0 < f(u) \leq c + \varepsilon_0 < 2^{\frac{4r-a}{2N-a}} S_{h,l}. \tag{5.14}
\]
On the other hand, Lemma 4.8 together with (5.8) gives
\[
\frac{S_{h,l} + c_0}{2} < c_0 - \varepsilon_0 \leq c - \varepsilon_0 < f(u). \tag{5.15}
\]
Now, (5.13) follows from (5.14) and (5.15).
Consider $u \in (\Sigma \cap M)$ with
\[
f(u) < \frac{S_{h,l} + c_0}{2}. \tag{5.16}
\]
Then,
\[
\tilde{h}(u) = u
\]
and from (5.16), we have that $u \notin f^{2(4s-a)(2N-a)S_{h,l}} \setminus f(S_{h,l} + c_0)/2$ and by (5.13), we have
\[
u \notin f^{c + \varepsilon_0 \setminus f^{c - \varepsilon_0}}.
\]
Thus,
\[ u \in f^{c-\varepsilon_0} \cup \{ (\Sigma \cap \mathcal{M}) \setminus f^{c+\varepsilon_0} \} \]
and by Deformation Lemma, we obtain
\[ \eta(1, u) = u. \]

Therefore,
\[ h^*(u) = \eta(1, \tilde{h}(u)) = \eta(1, u) = u, \]
which shows that \( h^* \in \mathcal{H} \), and so,
\[ h^*(\Theta) = \eta(1, \tilde{h}(\Theta)) = \eta(1, \tilde{A}) \in \Gamma. \] (5.17)

Hence,
\[ c = \inf_{\mathcal{A} \in \Gamma} \max_{u \in \mathcal{A}} f(u) \leq \max_{u \in \eta(1, \tilde{A})} f(u). \] (5.18)

On the other hand, from Deformation Lemma and (5.10), we get
\[ \eta(1, \tilde{A}) \subset \eta(1, f^{c+\frac{\eta}{2}}) \subset f^{c-\frac{\varepsilon_0}{2}}. \]
That is,
\[ f(u) \leq c - \frac{\varepsilon_0}{2}, \quad \forall u \in \eta(1, \tilde{A}), \]
which implies that
\[ \max_{u \in \eta(1, \tilde{A})} f(u) \leq c - \frac{\varepsilon_0}{2} \]
and using (5.18), we conclude that
\[ c \leq \max_{u \in \eta(1, \tilde{A})} f(u) \leq c - \frac{\varepsilon_0}{2}, \]
which is a contradiction. Therefore, we must have
\[ f(u_n) \to c \quad \text{and} \quad f'|_{\mathcal{M}}(u_n) \to 0 \]
and from Lemma 3.6, up to a subsequence, we have \( u_n \to \tilde{u}_0 \) in \( D^{s,2}(\mathbb{R}^N) \), and \( \tilde{u}_0 \) satisfies that
\[ f(\tilde{u}_0) = c \quad \text{and} \quad f'|_{\mathcal{M}}(\tilde{u}_0) = 0, \]
and from (5.7), we get \( S_{h,l} < f(\tilde{u}_0) < 2^{(4s-\alpha)(2N-\alpha)}S_{h,l} \). The positivity of \( \tilde{u}_0 \) is a consequence of a maximum principle that can be found in Proposition 2.17 [49]. □
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Appendix

As we pointed out in Introduction, the nonlocal term is involved with critical growth, and we shall encounter the problem of the convergence of integral with nonlocal term. To this end, we present a technical lemma which is useful in the proof of Lemma 3.1 and Theorem 3.2.

Lemma A. Let $(\xi_n)$ be a bounded sequence in $D^{s,2}(\mathbb{R}^N)$ such that $\xi_n \to 0$ a.e. in $\mathbb{R}^N$. Denote by $A(u) = (I_\alpha * |u|^{2_{\alpha,s}})|u|^{2_{\alpha,s}-2}u$. Then for each $w \in D^{s,2}(\mathbb{R}^N)$, we have the following estimates:

$$\int_{\mathbb{R}^N} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2_{\alpha,s}}{2_{\alpha,s}-1}} dx = o_n(1).$$

Proof. By assumption, we have

$$|A(\xi_n + w) - A(\xi_n)|$$

$$= \left| \int_0^1 \frac{d}{dt} \left( I_\alpha * |\xi_n + tw|^{2_{\alpha,s}}|\xi_n + tw|^{2_{\alpha,s}-2}[\xi_n + tw]\right) dt \right|$$

$$= \left| \int_0^1 (I_\alpha * (2_{\alpha,s}^*|\xi_n + tw|^{2_{\alpha,s}-2}[\xi_n + tw]w))|\xi_n + tw|^{2_{\alpha,s}-2}[\xi_n + tw]dt \right|$$

$$+ (2_{\alpha,s}^* - 1) \int_0^1 (I_\alpha * |\xi_n + tw|^{2_{\alpha,s}})|\xi_n + tw|^{2_{\alpha,s}-2}w dt$$

$$\leq 2_{\alpha,s}^* \int_0^1 (I_\alpha * (|\xi_n + tw|^{2_{\alpha,s}-1}|w|))|\xi_n + tw|^{2_{\alpha,s}-1}dt$$

$$+ (2_{\alpha,s}^* - 1) \int_0^1 (I_\alpha * |\xi_n + tw|^{2_{\alpha,s}})|\xi_n + tw|^{2_{\alpha,s}-2}|w| dt$$

$$\leq C_1 (I_\alpha * (|\xi_n|^{2_{\alpha,s}-1} + |w|^{2_{\alpha,s}-1}|w|))(|\xi_n|^{2_{\alpha,s}-1} + |w|^{2_{\alpha,s}-1})$$

$$+ C_2 (I_\alpha * (|\xi_n|^{2_{\alpha,s}} + |w|^{2_{\alpha,s}})(|\xi_n|^{2_{\alpha,s}-2} + |w|^{2_{\alpha,s}-2})|w|$$

$$\leq C_1 (I_\alpha * (\varepsilon|\xi_n|^{2_{\alpha,s}} + C_\varepsilon |w|^{2_{\alpha,s}}))(|\xi_n|^{2_{\alpha,s}-1} + |w|^{2_{\alpha,s}-1})$$
\begin{align*}
&+ C_2 (I_\alpha \ast (|\xi_n|^{2\alpha,s} + |w|^{2\alpha,s}) (|\xi_n|^{2\alpha,s} - 1 + C_\varepsilon |w|^{2\alpha,s-1}) \\
&\leq \varepsilon C_3 \left[ (I_\alpha \ast |\xi_n|^{2\alpha,s}) |\xi_n|^{2\alpha,s} - 1 + (I_\alpha \ast |w|^{2\alpha,s}) |\xi_n|^{2\alpha,s} - 1 + (I_\alpha \ast |\xi_n|^{2\alpha,s}) |w|^{2\alpha,s} - 1 \right] \\
&+ C_\varepsilon C_4 (I_\alpha \ast |\xi_n|^{2\alpha,s}) |w|^{2\alpha,s} - 1 + (I_\alpha \ast |w|^{2\alpha,s}) |\xi_n|^{2\alpha,s} - 1 \\
&+ C_\varepsilon C_5 (I_\alpha \ast |w|^{2\alpha,s}) |w|^{2\alpha,s} - 1 \\
&= Q_{\varepsilon,n}(x) + C_\varepsilon C_5 (I_\alpha \ast |w|^{2\alpha,s}) |w|^{2\alpha,s} - 1. \quad (A.1)
\end{align*}

Recall the Hardy-Littlewood-Sobolev inequality [36, Theorem 4.3]: if $\theta \in \left(1, \frac{N}{N-\alpha}\right)$ then for every $v \in L^\theta(\mathbb{R}^N)$, $I_\alpha \ast v \in L^{\frac{N\theta}{N-(N-\alpha)\theta}}(\mathbb{R}^N)$ and

\begin{equation}
\int_{\mathbb{R}^N} |I_\alpha \ast v|^{\frac{N\theta}{N-(N-\alpha)\theta}} dx \leq C \left( \int_{\mathbb{R}^N} |v|^\theta dx \right)^{\frac{N}{N-(N-\alpha)\theta}}. \quad (A.2)
\end{equation}

For each $\varepsilon > 0$, let us consider the function $G_{\varepsilon,n}$ given by

$$G_{\varepsilon,n}(x) = \max\{|A(\xi_n + w) - A(\xi_n) - A(w)|, 0\},$$

which satisfies

$$G_{\varepsilon,n}(x) \to 0 \text{ a.e. in } \mathbb{R}^N$$

and using Hölder’s inequality and (A.2) we see that

$$0 \leq G_{\varepsilon,n}(x) \leq C_6 (I_\alpha \ast |w|^{2\alpha,s}) |w|^{2\alpha,s} - 1 \in L^{\frac{2\alpha}{2\alpha-1}}(\mathbb{R}^N).$$

Therefore, by the Lebesgue dominated convergence theorem we have

\begin{equation}
\int_{\mathbb{R}^N} |G_{\varepsilon,n}(x)|^{\frac{2\alpha}{2\alpha-1}} dx \to 0 \text{ as } n \to \infty. \quad (A.3)
\end{equation}

From the definition of $G_{\varepsilon,n}(x)$, we get

$$|A(\xi_n + w) - A(\xi_n) - A(w)| \leq Q_{\varepsilon,n}(x) + C_7 G_{\varepsilon,n}(x),$$

which yields that

$$|A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2\alpha}{2\alpha-1}} \leq |Q_{\varepsilon,n}(x)|^{\frac{2\alpha}{2\alpha-1}} + C_8 |G_{\varepsilon,n}(x)|^{\frac{2\alpha}{2\alpha-1}}.$$
\[
\int_{\mathbb{R}^N} |A(\xi_n + w) - A(\xi_n) - A(w)|^{\frac{2^*}{2^* - 1}} \, dx \\
\leq \int_{\mathbb{R}^N} |Q_{e,n}(x)|^{\frac{2^*}{2^* - 1}} \, dx + C_8 \int_{\mathbb{R}^N} |G_{e,n}(x)|^{\frac{2^*}{2^* - 1}} \, dx. \quad (A.4)
\]

Now we estimate the last integral of (A.4). Since \((\xi_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\), \(\xi_n \to 0\) a.e. in \(\mathbb{R}^N\), then \(\xi_n \rightharpoonup 0\) in \(L^{2^*_s}(\mathbb{R}^N)\). By the definition of \(Q_{e,n}(x)\), we have

\[
\int_{\mathbb{R}^N} |Q_{e,n}(x)|^{\frac{2^*}{2^* - 1}} \, dx \\
\leq \varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |\xi_n|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx \\
+ \varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |w|^{2^*_s}) |\xi_n|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx + \varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |w|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx \\
+ C_\varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |\xi_n|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx + C_\varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |w|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx \\
= \varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |\xi_n|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx \\
+ D_\varepsilon \int_{\mathbb{R}^N} \left[ (I_s \ast |w|^{2^*_s}) |\xi_n|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx \\
+ D_\varepsilon \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |w|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1}} \, dx \\
= I_1 + I_2 + I_3, \quad (A.5)
\]

where \(D_\varepsilon = C(\varepsilon + C_\varepsilon)\). Next we estimate the three integrals in the right-side of (A.5).

For \(I_1\), by Hölder’s inequality, (A.2) and \((\xi_n)\) is bounded in \(D^{s,2}(\mathbb{R}^N)\), we have

\[
I_1 = \varepsilon C \int_{\mathbb{R}^N} \left[ (I_s \ast |\xi_n|^{2^*_s}) |\xi_n|^{2^*_s - 1} \right]^{\frac{2^*}{2^* - 1} \times \frac{2^*}{2^* - 1}} \, dx
\]
\[ \begin{align*}
&= \varepsilon C \int_{\mathbb{R}^N} (I_\alpha \ast |\xi_n|^\frac{2^* - a}{N + 2^*}) \frac{2N}{N + 2^*} |\xi_n|^\frac{N + 2^* - a}{N + 2^*} \frac{2N}{N + 2^*} dx \\
&\leq \varepsilon C \left[ \int_{\mathbb{R}^N} (I_\alpha \ast |\xi_n|^\frac{2^* - a}{N + 2^*}) \frac{2N}{N + 2^*} dx \right] \frac{2N}{N + 2^*} \left[ \int_{\mathbb{R}^N} |\xi_n|^\frac{N + 2^* - a}{N + 2^*} dx \right] \\
&\leq \varepsilon C \left[ \int_{\mathbb{R}^N} |\xi_n|^\frac{2^*}{N + 2^*} dx \right] \frac{2N}{N + 2^*} \left[ \int_{\mathbb{R}^N} |\xi_n|^\frac{2^*}{N + 2^*} dx \right] \\
&\leq \varepsilon C_1 \left[ \int_{\mathbb{R}^N} |(\Delta)^{\frac{1}{2}} \xi_n|^2 \right] \\
&\leq \varepsilon C_2. \quad (A.6)
\end{align*} \]

For \( I_2 \), we have
\[ I_2 = D_\varepsilon \int_{\mathbb{R}^N} (I_\alpha \ast |w|^\frac{2^*}{N + 2^*}) \frac{2N}{N + 2^*} |\xi_n|^\frac{N + 2^* - a}{N + 2^*} \times \frac{2N}{N + 2^*} dx \]
\[ = D_\varepsilon \int_{\mathbb{R}^N} (I_\alpha \ast |w|^\frac{2^*}{N + 2^*}) \frac{2N}{N + 2^*} |\xi_n|^\frac{N + 2^* - a}{N + 2^*} \times \frac{2N}{N + 2^*} dx \quad (A.7) \]
\[ = o_n(1), \]
by virtue of \( (I_\alpha \ast |w|^\frac{2^*}{N + 2^*}) \frac{2N}{N + 2^*} \in L^\frac{N+2^*}{\alpha} (\mathbb{R}^N) \) and \(|\xi_n|^\frac{N + 2^* - a}{N + 2^*} \times \frac{2N}{N + 2^*} \to 0 \) in \( L^\frac{N+2^*}{\alpha} (\mathbb{R}^N) \).

For \( I_3 \), we have
\[ I_3 = D_\varepsilon \int_{\mathbb{R}^N} (I_\alpha \ast |\xi_n|^\frac{2^*}{N + 2^*}) \frac{2N}{N + 2^*} \frac{N + 2^* - a}{N - 2^*} \times \frac{2N}{N + 2^*} \times \frac{N + 2^* - a}{N + 2^*} dx \]
\[ = D_\varepsilon \int_{\mathbb{R}^N} (I_\alpha \ast |\xi_n|^\frac{2^*}{N + 2^*}) \frac{2N}{N + 2^*} |w|^\frac{N + 2^* - a}{N - 2^*} \times \frac{2N}{N + 2^*} dx \quad (A.8) \]
\[ = o_n(1), \]
by virtue of \( (I_\alpha \ast |\xi_n|^\frac{2^*}{N + 2^*}) \frac{2N}{N + 2^*} \to 0 \) in \( L^\frac{N+2^*}{\alpha} (\mathbb{R}^N) \) and \(|w|^\frac{N + 2^* - a}{N - 2^*} \times \frac{2N}{N + 2^*} \in L^\frac{N+2^*}{\alpha} (\mathbb{R}^N) \). From (A.4) \(- (A.8)\), we obtain
\[ \int_{\mathbb{R}^N} |A(\xi_n + w) - A(\xi_n) - A(w)|^\frac{2^*}{N - 1} dx \to 0 \quad \text{as} \quad n \to \infty, \]
which completes the proof. \( \square \)
References


