Remarks on a polyharmonic eigenvalue problem

Remarques sur un problème poly-harmonique de valeurs propres

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Abstract

This Note deals with a nonlinear eigenvalue problem involving the polyharmonic operator on a ball in \( \mathbb{R}^n \). The main result of this Note establishes the existence of a continuous spectrum of eigenvalues such that the least eigenvalue is isolated.

Résumé

On considère un problème non linéaire de valeurs propres associé à l’opérateur polyharmonique sur une boule dans \( \mathbb{R}^n \). Dans cette Note on montre l’existence d’un spectre continu de valeurs propres tel que la valeur propre principale est isolée.

Let \( B \) be a ball of radius \( R \) in \( \mathbb{R}^n \) and let \( K \) be an integer strictly positive. In this Note we study the nonlinear eigenvalue problem

\[
\begin{aligned}
(-\Delta)^K u &= \lambda f(x, u) & \text{in } B, \\
u &= Du = \cdots = D^{K-1} u = 0 & \text{on } \partial B.
\end{aligned}
\]

On suppose que \( \lambda \) est un paramètre positif et que la fonction \( f \) est définie par

\[
f(x, t) = \begin{cases} t, & \text{si } t < 0, \\
h(x, t), & \text{si } t \geq 0,
\end{cases}
\]

where \( h : B \times \mathbb{R}^+ \to \mathbb{R} \) is a Carathéodory function such that \( H(x, t) := \int_0^t h(x, s) \, ds \), then the following conditions are satisfied:

\( (H_1) \) \quad \text{there exists } c \in (0, 1) \text{ such that } |h(x, t)| \leq ct \text{ for all } t \in \mathbb{R} \text{ and p.p. } x \in B; \\
(\text{H}_2) \quad \text{there exist } t_0 > 0 \text{ such that } H(x, t_0) > 0 \text{ for p.p. } x \in B; \\
(\text{H}_3) \quad \lim_{t \to \infty} \frac{h(x, t)}{t} = 0 \text{ uniformly on } B \setminus \bar{C}, \text{ where } \mu(C) = 0.
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On démontre que les valeurs de $\lambda$ pour lesquelles le problème (1) admet une solution sont liées à la première valeur propre du problème linéaire
\[
\begin{align*}
(-\Delta)^K u &= \lambda u \\
|u| &= D u = \cdots = D^{K-1} u = 0 & \text{in } B,
\end{align*}
\]
(3)

et il existe $\mu_1 > \lambda_1$ tel que chaque $\lambda \in (\mu_1, \infty)$ est une valeur propre du problème (1).

Les étapes principales dans la démonstration de ce résultat sont les suivantes :

(i) si $\lambda > 0$ est une valeur propre associée au problème (1), alors $\lambda \geq \lambda_1$ ;

(ii) la première valeur propre $\lambda_1$ du problème linéaire (3) est aussi une valeur propre du problème non linéaire (1) et, de plus, l'ensemble associé de fonctions propres est un cône dans l'espace de Hilbert $H^k_B (B)$ muni du produit scalaire
\[
(u, v)_K = \begin{cases}
\int_B (\Delta^m u) (\Delta^m v) \, dx, & \text{si } K = 2m, \\
\int_B (D^{\Delta^m} u) (D^{\Delta^m} v) \, dx, & \text{si } K = 2m + 1;
\end{cases}
\]

(iii) $\lambda_1$ est isolée dans l'ensemble de valeurs propres du problème (1);

(iv) il existe $\lambda^* > 0$ tel que $\inf_{H^k_B (B)} I_2 (u) < 0$ pour tout $\lambda \geq \lambda^*$, où
\[
I_2 (u) := \frac{1}{2} \| u \|^2_K - \lambda \int_B H (x, u, \partial u) \, dx, \quad u \in H^k_B (B)
\]
est l'énergie associée au problème (1).

1. Introduction

Let $B$ be any ball of $\mathbb{R}^n$ centered at the origin and of fixed radius $R > 0$. Consider the linear eigenvalue problem
\[
\begin{align*}
(-\Delta)^K u &= \lambda u \\
|u| &= D u = \cdots = D^{K-1} u = 0 & \text{in } B,
\end{align*}
\]
(4)

where $K$ is a positive integer. Then the lowest eigenvalue $\lambda_1$ of problem (4) is simple, that is, the associated eigenfunctions are merely multiples of each other. Moreover they are radial, strictly monotone in $r = |x|$ and never change sign in $B$. We refer to Pucci and Serrin [3] for further properties of eigenvalues of polyharmonic operators.

In this paper we are concerned with the nonlinear eigenvalue problem
\[
\begin{align*}
(-\Delta)^K u &= \lambda f (x, u) \\
|u| &= D u = \cdots = D^{K-1} u = 0 & \text{in } B,
\end{align*}
\]
(5)

where $\lambda$ is a positive parameter and the nonlinear function $f$ is given by
\[
f (x, t) = \begin{cases}
t, & \text{if } t < 0, \\
h (x, t), & \text{if } t \geq 0,
\end{cases}
\]
(6)

where $h : B \times \mathbb{R}_0^+ \to \mathbb{R}$ is a Carathéodory function, $H (x, t) := \int_0^t h (x, s) \, ds$, and the following conditions are fulfilled:

(H1) There exists $c \in (0, 1)$ such that $|h (x, t)| \leq c t$ for all $t \in \mathbb{R}$ and a.a. $x \in B$;

(H2) There exists $t_0 > 0$ such that $H (x, t_0) > 0$ for a.a. $x \in B$;

(H3) $\lim_{t \to \infty} \frac{h (x, t)}{t} = 0$ uniformly in $B \setminus \mathcal{O}$, with $\mu (\mathcal{O}) = 0$.

As already highlighted in [2], functions $h$ verifying (H1)–(H3) are given in $B \times \mathbb{R}_0^+$, e.g., by $h (x, t) = \sin (ct)$, $h (x, t) = c \log (1 + t)$, $h (x, t) = g (x) [t^{q (x)} - t^{p (x)}]$ where $c \in (0, 1)$, $p, q : \overline{B} \to (1, 2)$ continuous in $\overline{B}$, $\max_{\overline{B}} p (x) < \min_{\overline{B}} q (x)$, $g \in L^\infty (B)$, $\| g \|_\infty = c$. For the relevance of these examples in applications, as well as for a wide list of references, we refer to [2].

The main result of this Note is the following:
Theorem 1.1. Suppose that $f$ is of type (6) and that (H1)–(H3) are fulfilled. Then the first eigenvalue $\lambda_1$ of (4) is an isolated eigenvalue of problem (5) and the corresponding set of eigenfunctions is a cone. Moreover, any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of (5), while there exists $\mu_1 > \lambda_1$ such that any $\lambda \in (\mu_1, \infty)$ is an eigenvalue of (5).

2. Proof of Theorem 1.1

Consider the standard higher order Hilbertian Sobolev space $H^K_0 = H^K_0(B)$, endowed with the scalar product

$$(u, v)_K = \begin{cases} \int_B (\Delta^m u)(\Delta^m v) \, dx, & \text{if } K = 2m, \\ \int_B (D\Delta^m u)(D\Delta^m v) \, dx, & \text{if } K = 2m + 1, \end{cases}$$

and denote by $\| \cdot \|_K$ the corresponding norm. As in [1, Section 3], the decomposition method of Moreau and the comparison principle of Boggio in $H^K_0$ substitute the decomposition in the positive and negative part which is no longer admissible in the higher order Sobolev spaces when $K \geq 2$. Indeed, for any $u \in H^K_0$ there exists a unique couple $(u_1, u_2) \in \mathcal{K} \times \mathcal{K}'$ such that $u = u_1 + u_2$ and $(u_1, u_2)_K = 0$, where $\mathcal{K}$ is the convex closed cone of positive functions

$$\mathcal{K} = \{ v \in H^K_0 : v(x) \geq 0 \text{ a.e. in } B \},$$

while $\mathcal{K}'$ is the dual cone of $\mathcal{K}$, that is

$$\mathcal{K}' = \{ w \in H^K_0 : (w, v)_K \leq 0 \text{ for all } v \in \mathcal{K} \}.$$

By [1, Lemma 2] we know that $\mathcal{K}'$ is contained in the cone of negative functions, in other words $w(x) \leq 0$ a.e. in $B$ if $w \in \mathcal{K}'$. The number $\lambda > 0$ is an eigenvalue of problem (5), with $f$ of the type (6), if there exists $u \in H^K_0 \setminus \{0\}$ such that

$$(u, v)_K = \lambda \int_B f(x, u)v \, dx$$

for any $v \in H^K_0$.

Lemma 2.1. If $\lambda > 0$ is an eigenvalue of (5), then $\lambda \geq \lambda_1$.

Proof. Assume that $\lambda > 0$ is and eigenvalue of (5), with corresponding eigenfunction $u \in H^K_0 \setminus \{0\}$. Letting $v = u$ in (7), and putting $B_- = \{ x \in B : u(x) \leq 0 \}$ and $B_+ = \{ x \in B : u(x) \geq 0 \}$, we get by (H1)

$$\|u\|^2_K = \lambda \left[ \int_{B_+} h(x, u)u \, dx + \int_{B_-} u^2 \, dx \right] \leq \lambda \left[ c \int_{B_+} u^2 \, dx + \int_{B_-} u^2 \, dx \right] \leq \lambda |u|^2_2,$$

being $c \in (0, 1)$. By the definition of $\lambda_1$

$$\lambda_1 |u|^2_2 \leq \|u\|^2_K \leq \lambda |u|^2_2.$$

Since $u \neq 0$, then the above inequality shows that $\lambda \geq \lambda_1$. $\square$

Lemma 2.2. The first eigenvalue $\lambda_1$ of (4) is also an eigenvalue of (5) and the set of the corresponding eigenfunctions is a cone of $H^K_0$.

Proof. As already noted in the introduction the lowest eigenvalue $\lambda_1$ of (4) is simple, so that there exists a first eigenfunction $\varphi \in H^K_0 \setminus \{0\}$, with $\varphi < 0$ in $B$. Hence $\varphi$ is an eigenfunction also of (5), since clearly satisfies (7) with $\lambda = \lambda_1$, being $(\varphi, v)_K = \lambda_1 \int_B \varphi v \, dx = \lambda_1 \int_B f(x, \varphi)v \, dx$ by (6). Moreover the set of the corresponding eigenfunctions lies in a cone of $H^K_0$. $\square$

Lemma 2.3. The first eigenvalue $\lambda_1$ of (4) is isolated in the set of eigenvalues of (5).

Proof. Let $\lambda > 0$ be an eigenvalue of (5) whose corresponding eigenfunction $u$ has Moreau's decomposition with $u_1 \neq 0$. Then, being $u_1 \in H^K_0$, we take $v = u_1$ in (7), and by the definition of $\lambda_1$ and (H1) we get

$$\lambda_1 |u_1|^2_2 \leq \|u_1\|^2_K = \lambda \left[ \int_{B_+} h(x, u_1)u_1 \, dx + \int_{B_-} u_1^2 \, dx \right] \leq \lambda c |u_1|^2_2.$$

Hence $\lambda \geq \lambda_1/c > \lambda_1$, being $c \in (0, 1)$. In particular, any eigenfunction $u$ corresponding to an eigenvalue $\lambda \in (0, \lambda_1 / c)$ has decomposition $u = u_2$, so that $u$ is also an eigenfunction of (4), since $u = u_2 \leq 0$ a.e. in $B$. It is known, as noted in the
introduction, that \( \lambda_1 < \lambda_2 \), where \( \lambda_2 \) is the second eigenvalue of (4). Hence any \( \lambda \in (\lambda_1, \delta) \), with \( \delta = \min[\lambda_1/c, \lambda_2] \), cannot be eigenvalue of (4) and in turn is not an eigenvalue of (5), by the argument above. This completes the proof. \( \square \)

As already noted, \( \lambda > 0 \) is an eigenvalue of the problem

\[
\begin{align*}
(\Delta)^K u &= \lambda h(x, u_+) \quad \text{in } B, \\
\lambda u &= Du = \cdots = D^{K-1} u = 0 \quad \text{on } \partial B,
\end{align*}
\]

(8)

if there exists \( u \in H_0^K \setminus \{0\} \) such that \( (u, v)_K = \lambda \int_B h(x, u_+) v \text{d}x \) for all \( v \in H_0^K \), that is if and only if \( u = 0 \) is a nontrivial critical point of the \( C^1 \) functional \( I_{\lambda} : H_0^K \to \mathbb{R} \), defined by

\[
I_{\lambda}(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2} \int_B H(x, u) \text{d}x.
\]

If \( \lambda > 0 \) is an eigenvalue of (8), with corresponding eigenfunction \( u = u_1 + u_2 \), then taking as test function \( v = u_2 \) by (H1) we get, being \( (u_1, u_2)_K = 0 \) and \( h(x, 0) = 0 \) a.e. in \( B \),

\[
\|u_2\|^2_2 = (u, u)_K = \lambda \int_B h(x, u_+) u_2 \text{d}x = \lambda \int_B h(x) u_2 \text{d}x \leq 0,
\]

being \( u_2 \leq 0 \) a.e. in \( B \), that is \( u = u_1 \geq 0 \) in \( B \) and \( u \neq 0 \). In particular, any eigenvalue \( \lambda \) of (8) is also an eigenvalue of (5).

Assumption (H2) implies that for every \( \lambda > 0 \) there exists \( C_\lambda > 0 \) such that \( \lambda h(x, t) \leq C_\lambda + \lambda_1 t^2/4 \) for a.a. \( x \in B \) and all \( t \in \mathbb{R} \), where \( \lambda_1 \) is the first eigenvalue of (4). Hence, by the definition of \( \lambda_1 \), we have that for all \( u \in H_0^K \)

\[
I_{\lambda}(u) \geq \frac{1}{2} \|u\|^2_2 - \frac{\lambda_1}{4} \|u\|^2_2 - C_\lambda |B| \geq \frac{1}{4} \|u\|^2_2 - C_\lambda |B|,
\]

in other words \( I_\lambda \) is bounded from below, weakly lower semi-continuous and coercive on \( H_0^K \).

Lemma 2.4. There exists \( \lambda^* > 0 \) such that \( \inf_{H_0^K} I_{\lambda}(u) < 0 \) for all \( \lambda > \lambda^* \).

Proof. By (H2) there exists \( t_0 > 0 \) such that \( h(x, t_0) > 0 \) a.e. in \( B \). Let \( \Omega \subset B \) be a compact subset, sufficiently large, such that \( |B \setminus \Omega| \leq \int_B h(x, t_0) \text{d}x/c_\lambda^2 \), where \( c \in (0, 1) \) is given in (H1). Take \( u_0 \in C^\infty(B) \), with \( u_0(x) = t_0 \) if \( x \in \Omega \) and \( 0 \leq u_0(x) \leq t_0 \) if \( x \in B \setminus \Omega \). Hence, by (H1),

\[
\int_B H(x, u_0(x)) \text{d}x \geq c_\lambda^2 \int_B H(x, t_0) \text{d}x - c_\lambda^2 |B \setminus \Omega| > 0,
\]

and so \( I_\lambda(u_0) < 0 \) for \( \lambda > 0 \) sufficiently large. The lemma follows at once. \( \square \)

Now, we return to the proof of Theorem 1.1. Since \( I_\lambda \) is bounded from below, weakly lower semi-continuous and coercive on \( H_0^K \), then Lemma 2.3 and [4, Theorem 1.2] show that \( I_\lambda \) has a negative global minimum for \( \lambda > 0 \) sufficiently large. This means that all such \( \lambda \) are eigenvalues of problem (8) and, consequently, of (5). This fact and Lemmas 2.1–2.3 complete the proof of Theorem 1.1.

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References


