An infinity of nodal solutions for superlinear Robin problems with an indefinite and unbounded potential

Nikolaos S. Papageorgiou\textsuperscript{a,b}, Vicenţiu D. Rădulescu\textsuperscript{c,d,*}

\textsuperscript{a} National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece
\textsuperscript{b} King Saud University, Department of Mathematics, P.O. Box 2454, Riyadh 11451, Saudi Arabia
\textsuperscript{c} Department of Mathematics, Faculty of Sciences, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
\textsuperscript{d} Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

\textbf{Article history:}
Received 25 August 2016
Available online 5 April 2017

\textbf{MSC:}
35J20
35J60

\textbf{Keywords:}
Nodal solutions
Symmetric mountain pass theorem
C-condition
Regularity theory

\textbf{Abstract}
We consider a semilinear Robin problem driven by the Laplacian plus an indefinite and unbounded potential using a suitable version of the symmetric mountain pass theorem, we show that the problem has an infinity of nodal solutions whose energy level diverges to $+\infty$.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial \Omega$. In this paper, we study the following semilinear Robin problem

$$
\begin{align*}
-\Delta u(z) + \xi(z)u(z) &= f(z, u(z)) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(z)u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

(1)

In this problem, the potential function $\xi \in L^s(\Omega)$ with $s > N$ and is indefinite (that is, $\xi(\cdot)$ is sign changing). The reaction term $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ $z \to f(z, x)$ is measurable and for almost all $z \in \Omega$ $x \to f(z, x)$ is continuous), which is superlinear in the $x \in \mathbb{R}$ variable, but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short). In addition, for almost all $z \in \Omega$ $f(z, \cdot)$ satisfies a one-sided Lipschitz condition and it is odd. In the boundary condition $\frac{\partial u}{\partial n}$ denotes the usual normal derivative defined by extension of the map

$$
C^1(\partial \Omega) \ni u \to \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},
$$

with $n(\cdot)$ being the outward unit normal on $\partial \Omega$. The boundary coefficient $\beta \in W^{1,\infty}(\partial \Omega)$ with $\beta(z) \geq 0$ for all $z \in \partial \Omega$. When $\beta \equiv 0$, then we recover the Neumann problem.

We are looking for the existence of multiple nodal (that is, sign changing) solutions for problem (1). Using a version of the symmetric mountain pass theorem due to Qian and Li [13, Theorem 4.2], we show the existence of a sequence of distinct nodal solutions with energies diverging to $+\infty$.

In the past, an infinity of nodal solutions for superlinear Dirichlet problems with $\xi \equiv 0$, we proved by Qian and Li [13, Theorem 5.4] using AR-condition and with more restrictive conditions on the reaction term $f$. Subsequently, Qian [12, Theorem 1.1] produced an infinity of nodal solutions for a superlinear Neumann problem with $\xi \equiv a \in (0, +\infty)$. So, in Qian [12] the differential operator (right-hand side of the equation), is coercive and this simplifies the arguments considerably. Qian [12] did not use the AR-condition and instead employed a condition which was first introduced by Jeanjean [4]. This condition is global in nature and for this reason not entirely satisfactory. For Robin problems, there is the work of Qian and Li [14], who assume that $\xi \equiv 0$ and $f \in C(\bar{\Omega} \times \mathbb{R})$ satisfies the Jeanjean condition. They produce an infinity of distinct solutions, but they do not show that these solutions are nodal (see [14, Theorem 1.3]).

Problems with indefinite linear part (that is, the potential function $\xi(\cdot)$ is indefinite), were investigated by Zhang and Liu [18], Qin, Tang and Zhang [15], Zhang, Tang and Zhang [19]. All the aforementioned works deal with Dirichlet problems and use a nonquadraticity condition analogous to the one employed by Costa and Magalhaes [2]. They produce infinitely many nontrivial solutions, but the not show that they are nodal. Multiple nodal solutions for problems with indefinite linear part, were produced by Papageorgiou and Papalini [7] (Dirichlet problems), Papageorgiou and Rădulescu [8]
(Neumann problems) and Papageorgiou and Rădulescu [10] (Robin problems). None of the above works produces infinitely many nodal solutions. Finally we mention the very recent paper of Papageorgiou and Rădulescu [11], who produce a sequence of nodal solutions for nonlinear Robin problems but under different conditions and using different tools.

2. Mathematical background

Let $X$ be a Banach space and $X^*$ its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(X^*, X)$. Given $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi$ satisfies the “Cerami condition” (the “C-condition” for short), if the following property holds

“Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^*,$$

admits a strongly convergent subsequence”.

This is a compactness-type condition on $\varphi$, more general than the usual Palais–Smale condition. Nevertheless, it leads to the same deformation theorem from which one can derive the minimax theory of the critical values of $\varphi$.

The following spaces will be important in our analysis:

- The Sobolev space $H^1(\Omega)$;
- The Banach space $C^1(\overline{\Omega})$;
- The “boundary” Lebesgue spaces $L^q(\partial \Omega), \ 1 \leq q \leq \infty$.

The Sobolev space $H^1(\Omega)$ is a Hilbert space with inner product

$$\langle u, h \rangle_{H^1(\Omega)} = \int_{\Omega} uhdz + \int_{\Omega} (Du, Dh)_{\mathbb{R}^N}dz$$

for all $u, h \in H^1(\Omega)$ and corresponding norm

$$||u|| = [||u||^2 + ||Du||^2_{\mathbb{R}^N}]^{1/2}$$

for all $u \in H^1(\Omega)$.

The Banach space $C^1(\overline{\Omega})$ is an order Banach space with positive (order) cone given by

$$C_+ = \{ u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega} \}.$$ 

This cone has a nonempty interior containing

$$D_+ = \{ u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega} \}.$$
On \( \partial \Omega \) we consider the \( (N - 1) \)-dimensional Hausdorff (surface) measure \( \sigma(\cdot) \). Using this measure, we can define in the usual way the Lebesgue spaces \( L^q(\partial \Omega) \) \( 1 \leq q \leq \infty \). According to the theory of Sobolev spaces, there exists a unique continuous linear map \( \gamma_0 : H^1(\Omega) \to L^2(\partial \Omega) \) known as the “trace map”, which satisfies

\[
\gamma_0(u) = u|_{\partial \Omega} \text{ for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).
\]

So, the trace map assigns “boundary values” to all Sobolev functions. This map is compact into \( L^q(\partial \Omega) \) for all \( q \in \left[ 1, \frac{2N-2}{N-2} \right) \) if \( N \geq 3 \) and into \( L^q(\partial \Omega) \) for all \( q \geq 1 \) if \( N = 1, 2 \). In addition we have

\[
\ker \gamma_0 = H^1_0(\Omega) \text{ and } \text{im} \gamma_0 = H^{\frac{1}{2}, 2}(\partial \Omega).
\]

In the sequel, for the sake of notational economy, we drop the use of the map \( \gamma_0 \). All restrictions of Sobolev functions on \( \partial \Omega \) are understood in the sense of traces.

Next we consider the following linear eigenvalue problem:

\[
\begin{cases}
-\Delta u(z) + \xi(z)u(z) = \hat{\lambda}u(z) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(z)u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (2)

We impose the following conditions on the data of this eigenvalue problem

- \( \xi \in L^\infty(\Omega) \) if \( N \geq 3 \), \( \xi \in L^q(\Omega) \) with \( q > 1 \) if \( N = 2 \) and \( \xi \in L^1(\Omega) \) if \( N = 1 \).
- \( \beta \in W^{1,\infty}(\partial \Omega) \) with \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \).

Let \( \vartheta : H^1(\Omega) \to \mathbb{R} \) be the \( C^1 \)-functions defined by

\[
\vartheta(u) = \||Du||_2^2 + \int_{\Omega} \xi(z)u^2d\sigma + \int_{\partial \Omega} \beta(z)u^2d\sigma \text{ for all } u \in H^1(\Omega).
\]

We know (see [10]) that there exists \( \mu > 0 \) such that

\[
\vartheta(u) + \mu||u||_2^2 \geq c_0||u||^2 \text{ for all } u \in H^1(\Omega), \text{ some } c_0 > 0.
\] (3)

Using (3) and the spectral theorem for compact self-adjoint operators on a Hilbert space, we define the spectrum of (2) consisting of a sequence \( \{\hat{\lambda}_k\}_{k \geq 1} \subseteq \mathbb{R} \) such that \( \hat{\lambda}_k \to +\infty \). By \( E(\hat{\lambda}_k) \) \( k \in \mathbb{N} \) we denote the eigenspace corresponding to the eigenvalue \( \hat{\lambda}_k \). We know that each \( E(\hat{\lambda}_k) \) is finite dimensional and we have the following orthogonal direct sum decomposition

\[
H^1(\Omega) = \bigoplus_{k \geq 1} E(\hat{\lambda}_k).
\]
We know that

- \( \hat{\lambda}_1 \) is simple (that is, \( \dim E(\hat{\lambda}_1) = 1 \)).
- \( \hat{\lambda}_1 = \inf \left[ \frac{\vartheta(u)}{||u||_2^2} : u \in H^1(\Omega), u \neq 0 \right] \).
- \( \hat{\lambda}_m = \inf \left[ \frac{\vartheta(u)}{||u||_2^2} : u \in \bigoplus_{k \geq m} E(\hat{\lambda}_k), u \neq 0 \right] \)
  \[ = \sup \left[ \frac{\vartheta(u)}{||u||_2^2} : u \in \bigoplus_{k=1}^m E(\hat{\lambda}_k), u \neq 0 \right] \]
  \( m \geq 2 \).

The infimum in (4) is realized on \( E(\hat{\lambda}_1) \). Both the infimum and supremum in (5) are realized on \( E(\hat{\lambda}_m) \). Evidently the elements of \( E(\hat{\lambda}_1) \) do not change sign, while the elements of \( E(\hat{\lambda}_m) \) \( m \geq 2 \) are nodal (that is sign changing).

In what follows \( A : H^1(\Omega) \to H^1(\Omega)^* \) is the bounded linear operator defined by

\[ \langle A(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in H^1(\Omega). \]

Also, by \( | \cdot |_N \) we denote the Lebesgue measure on \( \mathbb{R}^N \) and \( 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2. \end{cases} \)

3. A sequence of nodal solutions

Our hypotheses on the data of (1) are the following:

- \( H(\xi) : \xi \in L^s(\Omega), s > N \) and \( \xi^+ \in L^\infty(\Omega) \).
- \( H(\beta) : \beta \in W^{1,\infty}(\partial \Omega) \) with \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \).
- \( H(f) : f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that for almost all \( z \in \Omega \) \( f(z, \cdot) \) is odd and

(i) \(|f(z,x)| \leq a(z)(1 + |x|^{r-1}) \) for almost all \( z \in \Omega \), all \( x \in \mathbb{R} \), with \( a \in L^\infty(\Omega), r \in (2, 2^*) \);
(ii) if \( F(z,x) = \int_0^x f(z,s)ds \), then \( \lim_{x \to \pm \infty} \frac{F(z,x)}{x} = +\infty \) uniformly for almost all \( z \in \Omega \);
(iii) if \( e(z,x) = f(z,x)x - 2F(z,x) \), then there exists \( d \in L^1(\Omega) \) such that

\[ e(z,x) \leq e(z,y) + d(z) \text{ for almost all } z \in \Omega, \text{ all } 0 \leq x \leq y \text{ or } y \leq x \leq 0; \]
(iv) there exists \( \hat{\eta} > 0 \) such that for almost all \( z \in \Omega \) the function

\[ x \to f(z,x) + \hat{\eta}x \]

is increasing on \( \mathbb{R} \).
(v) there exist \( \hat{c}_0, \hat{c}_1 > 0 \) such that

\[
-\hat{c}_0 \leq \liminf_{x \to 0} \frac{f(z,x)}{x} \leq \limsup_{x \to 0} \frac{f(z,x)}{x} \leq \hat{c}_1 \quad \text{uniformly for almost all } z \in \Omega.
\]

**Remark 1.** Hypothesis \( H(f)(ii) \) implies that the primitive \( F(z,\cdot) \) is superquadratic near \(+\infty\). Hypotheses \( H(f)(ii),(iii) \) imply that

\[
\lim_{x \to \pm\infty} \frac{f(z,x)}{x} = +\infty \quad \text{uniformly for almost all } z \in \Omega.
\]

So, the reaction term \( f(z,\cdot) \) is superlinear. However, this superlinearity is not expressed via the classical AR-condition, which says that there exist \( q > 2 \) and \( M > 0 \) such that

\[
0 < qF(z,x) \leq f(z,x)x \quad \text{for almost all } z \in \Omega, \; \text{all } |x| \geq M \quad (6a)
\]

\[
0 < \text{ess inf}_\Omega F(\cdot,\pm M) \quad (6b)
\]

(see Ambrosetti and Rabinowitz [1] and Mugnai [6]). Integrating \((6a)\) and using \((6b)\), we obtain the weaker condition

\[
c_1 |x|^q \leq F(z,x) \quad \text{for almost all } z \in \Omega, \; \text{all } |x| \geq M, \; \text{some } c_1 > 0
\]

This means that under the AR-condition \( f(z,\cdot) \) has at least \((q-1)\)-polynomial growth near \( \pm\infty \). The Jeanjean condition used in some works mentioned in the Introduction, says that there exist \( \eta \geq 1 \) and \( s \in [0,1] \) such that

\[
e(z, sx) \leq \eta e(z,x) \quad \text{for almost all } z \in \Omega, \; \text{all } x \in \mathbb{R}
\]

We mention the global nature of this condition. This is a feature which we would like to avoid. Here instead of the AR-condition and the Jeanjean condition we employ a quasimonotonicity condition on \( e(z,\cdot) \) (see hypothesis \( H(f)(iii) \)). This condition is a slightly more general version of a condition used by Li and Yang [5]. It is satisfied if there exists \( M \geq 0 \) such that

\[
e(z,\cdot) \quad \text{is nondecreasing on } x \geq M \quad \text{and nonincreasing on } x \leq -M.
\]

In turn, this is implied by the following condition

\[
x \to \frac{f(z,x)}{x} \quad \text{is nondecreasing on } x \geq M,
\]

\[
x \to \frac{f(z,x)}{x} \quad \text{is nonincreasing on } x \leq -M
\]
We stress the local character of the last two conditions. Hypothesis $H(f)(iv)$ is a one-sided Lipschitz condition. Finally hypothesis $H(f)(v)$ implies that for almost all $z \in \Omega$, $f(z, \cdot)$ is linear near zero.

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$
\varphi(u) = \frac{1}{2} \vartheta(u) - \int_\Omega F(z, u) dz \text{ for all } u \in H^1(\Omega).
$$

Evidently $\varphi \in C^1(H^1(\Omega))$.

**Proposition 1.** If hypotheses $H(\xi), H(\beta), H(f)$ hold, then the functional $\varphi$ satisfies the C-condition.

**Proof.** We consider a sequence $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ such that

\begin{align}
|\varphi(u_n)| & \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \quad (7) \\
(1 + |u|)\varphi'(u_n) & \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty \quad (8)
\end{align}

From (8) we have

\begin{align}
\left| \langle A(u_n), h \rangle + \int_\Omega \xi(z)u_n h dz + \int_{\partial \Omega} \beta(z)u_n h d\sigma - \int_\Omega f(z, u_n) h dz \right| & \leq \frac{\epsilon_n |h|}{1 + |u_n|} \quad (9)
\end{align}

for all $h \in H^1(\Omega)$, with $\epsilon_n \to 0^+$.

In (9) we choose $h = u_n \in H^1(\Omega)$. Then

\begin{align}
\vartheta(u_n) + \int_\Omega f(z, u_n) u_n dz & \leq \epsilon_n \text{ for all } n \in \mathbb{N} \quad (10)
\end{align}

From (7), we have

\begin{align}
-\vartheta(u_n) - \int_\Omega 2F(z, u_n) dz & \leq 2M_1 \text{ for all } n \in \mathbb{N} \quad (11)
\end{align}

Adding (10) and (11), we obtain

\begin{align}
\int_\Omega e(z, u_n) dz & \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \in \mathbb{N}. \quad (12)
\end{align}

**Claim 1.** $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ is bounded.
We argue by contradiction. So, suppose that the Claim is not true. By passing to a suitable subsequence if necessary, we may assume that

$$||u_n|| \to \infty.$$  \hfill (13)

Let $$y_n = \frac{u_n}{||u_n||}$$, $$n \in \mathbb{N}$$. Then $$||y_n|| = 1$$ for all $$n \in \mathbb{N}$$ and so we may assume that

$$y_n \xrightarrow{z} y \text{ in } H^1(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^r(\Omega) \text{ and } L^2(\partial\Omega).$$ \hfill (14)

(note that we can always assume $$r \geq \frac{2s}{s-1}$$, see hypothesis H(f)(i)).

First assume that $$y \neq 0$$ and let $$\Omega_0 = \{z \in \Omega : y(z) \neq 0\}$$. We have $$|\Omega_0|_N > 0$$ and

$$|u_n(z)| \to +\infty \text{ for almost all } z \in \Omega_0.$$  \hfill (15)

Using (15) and Fatou’s lemma (it can be used on account of hypothesis H(f)(iii)), we have

$$\frac{1}{||u_n||^2} \int_{\Omega_0} F(z,u_n)dz \to +\infty \text{ as } n \to \infty.$$ \hfill (16)

Hypothesis h(f)(ii) implies that we can find $$M_3 > 0$$ such that

$$F(z,x) \geq 0 \text{ for almost all } z \in \Omega, \text{ all } |x| \geq M_3.$$ \hfill (17)

We have

$$\frac{1}{||u_n||^2} \int_{\Omega} F(z,u_n)dz = \frac{1}{||u_n||^2} \int_{\Omega_0} F(z,u_n)dz + \frac{1}{||u_n||^2} \int_{\Omega_0 \cap \{|u_n| \geq M_3\}} F(z,u_n)dz +$$

$$\frac{1}{||u_n||^2} \int_{\Omega_0 \cap \{|u_n| < M_3\}} F(z,u_n)dz \geq$$

$$\frac{1}{||u_n||^2} \int_{\Omega_0} F(z,u_n)dz - \frac{c_2}{||u_n||^2} \text{ for some } c_2 > 0, \text{ all } n \in \mathbb{N}$$

(see (17 and use hypothesis H(f)(i)),

$$\Rightarrow \lim_{n \to \infty} \frac{1}{||u_n||^2} \int_{\Omega} F(z,u_n)dz = +\infty \text{ (see (16)).}$$ \hfill (18)
On the other hand from (7) we have
\[
\frac{1}{||u_n||^2} \int_{\Omega} F(z, u_n)dz \leq \frac{M_1}{||u_n||^2} + \frac{1}{2}\vartheta(y_n) \text{ for all } n \in \mathbb{N}
\]
\[
\Rightarrow \frac{1}{||u_n||^2} \int_{\Omega} F(z, u_n)dz \leq M_4 \text{ for some } M_4 > 0, \text{ all } n \in \mathbb{N}
\]
(19)
(see hypotheses $H(\xi), H(\beta)$ and recall that $||y_n|| = 1, \text{ } n \in \mathbb{N}$)

Comparing (16) and (19), we reach a contradiction.
Next suppose that $y = 0$. Given $\tau > 0$, let
\[
v_n = (2\tau)^{1/2}y_n \text{ for all } n \in \mathbb{N}.
\]
We have
\[
v_n \to 0 \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega) \text{ (see (14) and recall that } y=0).\]

It follows that
\[
\int_{\Omega} F(z, v_n)dz \to 0 \text{ as } n \to \infty.
\]
(20)

From (13) we see that we can find $n_0 \in \mathbb{N}$ such that
\[
0 < (2\tau)^{1/2} \frac{1}{||u_n||} \leq 1 \text{ for all } n \geq n_0
\]
(21)
Choose $t_n \in [0, 1]$ such that
\[
\varphi(t_n u_n) = max[\varphi(tu) : 0 \leq t \leq 1] \text{ for all } n \in \mathbb{N}.
\]
(22)

Taking into account (21), we have
\[
\varphi(t_n u_n) \geq \varphi(v_n)
\]
\[
= \tau\vartheta(y_n) - \int_{\Omega} F(z, u_n)dz
\]
\[
\geq \tau[c_0 - \mu||y_n||_2^2] - \int_{\Omega} F(z, v_n)dz \text{ for all } n \geq n_0 \text{ (see (3)).}
\]
(23)

Passing to the limit as $n \to \infty$ in (23) and using (14) and (20) and recalling that $y = 0$, we obtain
\[
\liminf_{n \to \infty} \varphi(t_n u_n) \geq \tau c_0.
\]
But $\tau > 0$ is arbitrary. So, it follows that
\[ \varphi(t_n u_n) \to +\infty \text{ as } n \to \infty. \]  
(24)

We have
\[ \varphi(0) = 0 \text{ and } \varphi(u_n) \leq M_1 \text{ for all } n \in \mathbb{N} \text{ (see (7))}. \]

The from (24) we infer that we can find $n_2 \in \mathbb{N}$ such that
\[ t_n \in (0,1) \text{ for all } n \geq n_2 \]  
(25)

From (22) and (25) it follows that
\[ \frac{d}{dt} \varphi(tu_n) \bigg|_{t=t_n} = 0 \text{ for all } n \geq n_2, \]
\[ \Rightarrow \langle \varphi'(t_n u_n), t_n u_n \rangle = 0 \text{ for all } n \geq n_2 \text{ (by the chain rule)}, \]
\[ \Rightarrow \vartheta(t_n u_n) = \int_{\Omega} f(z, t_n u_n)(t_n u_n)dz \text{ for all } n \geq n_2. \]  
(26)

Hypothesis $H(f)(iii)$ and (25) imply that
\[ \int_{\Omega} e(z, tu_n)dz \]
\[ = \int_{\Omega} [e(z, tu_n^+) + e(z, -t_n u_n^-)]dz \text{ (note that } e(z, 0) = 0 \text{ for almost all } z \in \Omega) \]
\[ \leq \int_{\Omega} [e(z, u_n^+) + e(z, -u_n^-)]dz + ||d||_1 \text{ (see hypothesis } H(f)(iii)) \]
\[ = \int_{\Omega} e(z, u_n)dz + ||d||_1 \text{ for all } n \geq n_2. \]

Using this inequality in (26), we obtain
\[ 2\varphi(t_n u_n) \leq \int_{\Omega} e(z, u_n)dz + ||d||_1 \]
\[ \leq M_2 + ||d||_1 = M_5 \text{ for all } n \geq n_2 \text{ (see (12))}. \]  
(27)

Comparing (24) and (27), we have a contradiction.
This proves the Claim.
On account of the Claim, we may assume that

$$u_n \overset{w}{\rightharpoonup} u \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^2(\partial\Omega).$$

(28)

In (9) we choose \( h = -u \in H^1(\Omega) \), pass to the limit as \( n \to \infty \) and use (28) (recall \( r \geq \frac{2s}{s-1} \)). We obtain

$$\lim_{n \to \infty} \langle A(u_n), u_n - u \rangle = 0,$$

$$\Rightarrow \| Du \|_2 \to \| Du \|_2,$$

$$\Rightarrow u_n \to u \text{ in } H^1(\Omega) \text{ (by the Kadec–Klee property for Hilbert spaces, see (28))},$$

$$\Rightarrow \varphi \text{ satisfies the C-condition.}$$

For every \( m \in \mathbb{N} \), we define

$$Y_m = \bigoplus_{k=1}^m E(\hat{\lambda}_k) \text{ and } V_m = \bigoplus_{k \geq m} E(\hat{\lambda}_k).$$

Let

$$\beta_m = \sup[\| u \|_r : u \in V_m, \| u \| = 1]$$

(29)

As in the proof of Lemma 3.8 of Willem [17, p. 60], we show that

$$\beta_m \to 0^+ \text{ as } m \to +\infty.$$  

(30)

**Proposition 2.** If hypotheses \( H(\xi), H(\beta), H(f) \) hold, then we can find \( \{l_m\}_{m \in \mathbb{N}} \subseteq (0,+\infty) \) such that

$$\gamma_m = \inf \{ \varphi(u) : u \in V_m, \| u \| = l_m \} \to +\infty \text{ as } m \to \infty.$$  

**Proof.** Let \( u \in V_m \). We have

$$\varphi(u) = \frac{1}{2} \vartheta(u) - \int_{\Omega} F(z,u)dz$$

$$= \frac{1}{2} \vartheta(u) + \mu \frac{\| u \|_2^2}{2} - \frac{\mu}{2} \| u \|_2^2 - \int_{\Omega} F(z,u)dz \text{ (with } \mu > 0 \text{ as in (3))}$$

$$\geq \frac{c_0}{2} \| u \|_2^2 - \frac{\mu}{2} \| u \|_2^2 - c_3 \| u \|_r^2 - c_3 \text{ for some } c_3 > 0$$

(see (3 and hypothesis H(f)(i))

(31)

Recall that \( r > 2 \) (see hypothesis \( H(f)(i) \)). So, we can find \( c_4 > 0 \) such that

$$\| u \|_2 \leq c_4 \| u \|_r \text{ for all } u \in H^1(\Omega)$$

(32)
Using (32) and (31) we obtain
\[ \varphi(u) \geq \frac{c_0}{2} ||u||^2 - c_5 (||u||_r^r + ||u||_r^2) - c_3 \text{ for some } c_5 > 0, \text{ all } u \in V_m \]

Suppose that \( ||u|| \geq 1 \). Then using once more the fact that \( r > 2 \), we obtain
\[ \varphi(u) \geq \frac{c_0}{2} ||u||^2 - c_0 ||u||_r^r - c_3 \text{ with } c_0 = 2c_5 > 0, \text{ all } u \in V_m, ||u|| \geq 1 \]

From (29) we have
\[ \beta_m ||u|| \geq ||u||_r \text{ for all } u \in V_m \]

Using this inequality in (33), we obtain
\[ \varphi(u) \geq \frac{c_0}{2} ||u||^2 - c_0 \beta_m ||u||^r - c_3 \text{ for all } u \in V_m, ||u|| \geq 1. \]

Let \( l_m = \left( \frac{c_0}{c_0} r \beta_m \right)^{\frac{2}{r}} \), we have
\[ l_m \to +\infty \text{ as } m \to +\infty \text{ (see (30) and recall that } r > 2). \]

Hence we may assume that \( l_m \geq 1 \) for all \( m \in \mathbb{N} \). Then from (34) we see that for all \( u \in V_m \) with \( ||u|| = l_m \), we have
\[ \varphi(u) \geq \frac{c_0}{2} \left( \frac{c_0}{r} \beta_m \right)^{\frac{2}{r}} - c_0 \beta_m \left( \frac{c_0}{r} \beta_m \right)^{\frac{2}{r}} \\
= c_0 \left[ \frac{1}{2} - \frac{1}{r} \right] \left( \frac{c_0}{r} \beta_m \right)^{\frac{2}{r}} , \\
\Rightarrow l_m \to +\infty \text{ as } m \to +\infty \text{ (see (30) and recall that } r > 2). \]

**Proposition 3.** If hypotheses \( H(\xi), H(\beta), H(f) \) hold, then we can find \( \{\rho_m\}_{m \in \mathbb{N}} \subseteq (0, \infty) \), \( \rho_0 > \lambda > 0 \) for all \( m \in \mathbb{N} \) such that
\[ \mathcal{S}_m = \sup_{u \in Y_m} [\varphi(u) : u \in Y_m, ||u|| = \rho_m] \leq 0 \text{ for all } m \in \mathbb{N}. \]

**Proof.** Let \( u \in Y_m \). We have
\[ \varphi(u) = \frac{1}{2} \partial(u) - \int_{\Omega} F(z, u) dz \]
\[ \leq \frac{1}{2} ||Du||_2^2 + \frac{1}{2} \int_{\Omega} \xi^+(z)u^2 dz + \frac{1}{2} \int_{\partial\Omega} \beta(z)u^2 d\sigma - \int_{\Omega} F(z, u) dz. \]
Hypotheses $H(f)(i), (ii)$ imply that given any $\eta > 0$, we can find $c_7 = c_7(\eta) > 0$ such that

$$F(z, x) \geq \eta x^2 - c_7$$

for almost all $z \in \Omega$, all $x \in \mathbb{R}$.

Using this unilateral growth estimate and hypothesis $H(\xi)$ in (35), we obtain

$$\varphi(u) \leq c_8 ||u||^2 - \eta ||u||^2 + c_7 |\Omega|_N$$

for some $c_8 > 0$, all $u \in Y_m$.

But $Y_m$ is finite dimensional. So, all norms are equivalent. Hence we can find $c_9 > 0$ such that

$$\varphi(u) \leq c_8 ||u||^2 - \eta c_9 ||u||^2 + c_7 |\Omega|_N$$

for all $u \in Y_m$. (36)

Recall that $\eta > 0$ is arbitrary. So, we choose $\eta > \frac{c_8}{c_9}$. Then from (36) it is clear that we can find $\rho_m > l_m$ $m \in \mathbb{N}$ such that

$$\varphi(u) \leq 0 \text{ for all } u \in Y_m, ||u|| = \rho_m,$$

$$\Rightarrow \mathfrak{Z}_m \leq 0 \text{ for all } m \in \mathbb{N}. \quad \square$$

**Proposition 4.** If hypotheses $H(\xi), H(\beta), H(f)$ hold and $u \in H^1(\Omega)$ is a solution of (1), then $u \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s} > 0$ (see hypothesis $H(\xi)$).

**Proof.** Hypotheses $H(f)(i), (v)$ imply that

$$|f(z, x)| \leq c_{10}(|x| + |x|^{r-1})$$

for almost all $z \in \Omega$, all $x \in \mathbb{R}$, some $c_{10} > 0$. (37)

By hypothesis we have

$$-\Delta u(z) + \xi(z) u(z) = f(z, u(z))$$

for almost all $z \in \Omega$

(see also Papageorgiou and Rădulescu [9]),

$$\Rightarrow -\Delta u(z) = \left[ \frac{f(z, u(z))}{u(z)} - \xi(z) \right] u(z)$$

for almost all $z \in \Omega$.

Note that $f(z, 0) = 0$ for almost all $z \in \Omega$ (see hypothesis $H(f)(v)$) and let

$$\hat{a}(z) = \begin{cases} \frac{f(z, u(z))}{u(z)} & \text{if } u(z) \neq 0 \\ 0 & \text{if } u(z) = 0 \end{cases}$$
Then

\[ |\hat{a}(z)| \leq \frac{|f(z, u(z))|}{|u(z)|} + |\xi(z)| \]

\[ \leq c_{10}(1 + |u(z)|^{r-2}) + |\xi(z)| \text{ for almost all } z \in \Omega \text{ (see (37))} \]

Note that \(|u(\cdot)|^{r-2} \in L^{\frac{2^*}{r-2}}(\Omega)\) (recall that \(u \in H^1(\Omega)\) and use the Sobolev embedding theorem) and observe that \(\frac{2^*}{r-2} > \frac{N}{2}\) (recall that \(r < 2^*\)). Therefore

\[ \hat{a} \in L^q(\Omega) \text{ with } q > \frac{N}{2} \text{ (see hypothesis } H(\xi)) \]

Then Lemma 5.1 of Wang [16] implies that

\[ u \in L^\infty(\Omega). \]

Using this fact and hypotheses \(H(f)(i)\) and \(H(\xi)\), we have

\[ f(\cdot, u(\cdot)) - \xi(\cdot)u(\cdot) \in L^s(\Omega) \text{ with } s > N. \]

So, the Calderon–Zygmund estimates (see Wang [16, Lemma 5.2]), we have

\[ u \in W^{2,s}(\Omega). \]

The Sobolev embedding theorem implies that

\[ u \in C^{1,\alpha}(\Omega) \text{ with } \alpha = 1 - \frac{N}{s} > 0. \]

**Proposition 5.** If hypotheses \(H(\xi), H(\beta), H(f)\) hold and \(u, v \in H^1(\Omega)\) are distinct solutions of (1) such that \(v \leq u\), then \(u - v \in D_+\).

**Proof.** From Proposition 4, we know that

\[ u, v \in C^1(\Omega). \]

Let \(\hat{\eta} > 0\) be as in hypothesis \(H(f)(iv)\). Then

\[ -\Delta v(z) + (\xi(z) + \hat{\eta})v(z) \]

\[ = f(z, v(z)) + \hat{\eta}v(z) \]

\[ \leq f(z, u(z)) + \hat{\eta}u(z) \text{ (see hypothesis } H(f)(iv) \text{ and recall that } v \leq u) \]

\[ = -\Delta u(z) + (\xi(z) + \hat{\eta})u(z) \text{ for almost all } z \in \Omega, \]
⇒ \Delta(u - v)(z) \leq [||\xi^+||_{\infty} + \eta](u - v)(z) \text{ for almost all } z \in \Omega \\
(\text{see hypothesis } H(\xi)), \\
⇒ u - v \in D_+ \\

(by the strong maximum principle, see Gasinski and Papageorgiou [3, p. 738]). □

**Corollary 6.** If hypotheses $H(\xi), H(\beta), H(f)$ hold and $u \in H^1(\Omega), u \neq 0, u \geq 0$ is a solution of (1), then $u \in D_+$.

From Proposition 5, Corollary 6 and Proposition 5.4 of Qian and Li [13] (see also the proof of Theorem 2 in Papageorgiou and Papalini [7]), we obtain the following result.

**Proposition 7.** If hypotheses $H(\xi), H(\beta), H(f)$ hold, then $C_+$ is an admissible invariant set of $\varphi$.

We make a final observation before formulating our multiplicity theorem.

**Proposition 8.** If hypotheses $H(\xi), H(\beta), H(f)$ and $l_m > 0 m \in \mathbb{N}$ is as in Proposition 2, then $V_m \cap \partial B_{l_m} \cap C_+ = \emptyset$ for all $m \geq 2$ (hence $\partial B_{l_m} = \{u \in H^1(\Omega) : ||u|| = l_m\}$).

**Proof.** Let $\hat{u}_1$ be the positive, $L^2$-normalized (that is, $||\hat{u}_1||_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. The regularity theory (see [16]) and the strong maximum principle (see [3]), imply that $\hat{u}_1 \in D_+$.

For $u \in C_+ \setminus \{0\}$ we have

$$\int_{\Omega} u \hat{u}_1 dz > 0$$

On the other hand for every $u \in V_m$ with $m \geq 2$, we have

$$\int_{\Omega} u \hat{u}_1 dz = 0 \text{ (since } V_m^1 \supseteq E(\hat{\lambda}_1)).$$

Therefore $V_m \cap \partial B_{l_m} \cap C_+ = \emptyset$. □

All these auxiliary results permit the use of Theorem 4.2 of Qian and Li [13] (the symmetric mountain pass theorem). So, we have the following multiplicity theorem.

**Theorem 9.** If hypotheses $H(\xi), H(\beta), H(f)$ hold, the problem (1) admits a sequence \(\{u_n\}_{n \geq 1} \subseteq C^1(\Omega)\) of distinct nodal solutions such that $\varphi(u_n) \to +\infty$.

**Conflict of interest statement**

There is no conflict of interest.
Acknowledgments

V. Rădulescu was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2016-0130.

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