Research Article

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Multiplicity and Concentration of Solutions for Kirchhoff Equations with Magnetic Field

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Abstract: In this paper, we study the following nonlinear magnetic Kirchhoff equation:

\[
\begin{cases}
-(a\epsilon^2 + b\epsilon[u]_{A/\epsilon}^2)\Delta_{A/\epsilon} u + V(x)u = f(|u|^2)u \\
u \in H^1(\mathbb{R}^3, \mathbb{C}),
\end{cases}
\]

where \( \epsilon > 0, a, b > 0 \) are constants, \( V : \mathbb{R}^3 \to \mathbb{R} \) is a continuous potential, and \( \Delta_{A/\epsilon} u \) is the magnetic Laplace operator. Under a local assumption on the potential \( V \), by combining variational methods, a penalization technique and the Ljusternik–Schnirelmann theory, we prove multiplicity properties of solutions and concentration phenomena for \( \epsilon \) small. In this problem, the function \( f \) is only continuous, which allows to consider larger classes of nonlinearities in the reaction.

Keywords: Kirchhoff Equation, Magnetic Field, Concentration, Multiple Solutions, Variational Methods

MSC 2010: 35J20, 35J60, 58E05

1 Introduction and Main Results

This paper is devoted to the qualitative analysis of solutions for the nonlinear magnetic Kirchhoff equation in \( \mathbb{R}^3 \). We are concerned with the existence and multiplicity of solutions, as well as with concentration properties of solutions for small values of the positive parameter. A feature of this paper is that the reaction has weak regularity, which allows to consider larger classes of nonlinearities. The main result is described in the final part of this section.

In this paper, we study the following nonlinear magnetic Kirchhoff equation:

\[
\begin{cases}
-(ae^2 + b\epsilon[u]_{A/\epsilon}^2)\Delta_{A/\epsilon} u + V(x)u = f(|u|^2)u \\
u \in H^1(\mathbb{R}^3, \mathbb{C}),
\end{cases}
\]

where \( \epsilon > 0, a, b > 0 \) are constants, \( V : \mathbb{R}^3 \to \mathbb{R} \) is a continuous function, the magnetic potential \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) is Hölder continuous with exponent \( a \in (0, 1] \), and \( -\Delta_A u \) is the magnetic Laplace operator of the following form:

\[
-\Delta_A u := \left( \frac{1}{i} \nabla - A(x) \right)^2 u = -\Delta u - \frac{2}{i} A(x) \cdot \nabla u + |A(x)|^2 u - \frac{1}{i} u \text{ div}(A(x)).
\]

The definition of \( [u]_{A/\epsilon}^2 \) will be given in Section 2.

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For problem (1.1), there is a vast literature concerning the existence and multiplicity of bound state solutions for the case $A \equiv 0$ and $a = b = 0$. The first result in this direction was given by Floer and Weinstein in [8], where the case $N = 1$ and $f = i_R$ is considered. Later on, several authors generalized this result to larger values of $N$, using different methods. For instance, He and Zou [10] considered the following fractional Schrödinger equation:

$$
\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u) + u^{2^*-1}, \quad x \in \mathbb{R}^N,
$$

where $V$ is a positive continuous function and satisfies the local assumption $\inf_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x)$, and $f \in C^1$ is a function having subcritical and superlinear growth. By using the Nehari manifold method and the Ljusternik–Schnirelman category theory, they obtained the multiplicity of positive solutions. We note that $f$ is only continuous, and the Nehari manifold is only a topological manifold. He and Zou [10] applied the method that Szulkin and Weth developed in [20]. He and Zou [11] also studied multiplicity of concentrating solutions for a class of fractional Kirchhoff equations when the potential satisfies a local assumption and the nonlinear term $f$ is only continuous. We also note that Ji, Fang and Zhang [12] considered a multiplicity result for asymptotically linear Kirchhoff equations. For further results about Kirchhoff equations, see [9, 19, 22, 23] and the references therein.

On the other hand, when $a = b = 0$, the magnetic nonlinear Schrödinger equation (1.1) has been extensively investigated by many authors applying suitable variational and topological methods (see [1, 3, 4, 7, 15, 16, 18, 24, 25] and the references therein). It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [7]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed and $N = 2, 3$. In particular, due to our scope, we want to mention [1] where Alves, Figueiredo and Furtado used the method of the Nehari manifold, the penalization method and Ljusternik–Schnirelman category theory for subcritical nonlinearity $f \in C^1$. We point out that if $f$ is only continuous, then the arguments developed in [1] fail. In [13, 14], Ji and Rădulescu used the method of the Nehari manifold, the penalization method and Ljusternik–Schnirelman category theory to study the multiplicity and concentration results for a magnetic Schrödinger equation in which the nonlinearity $f$ is only continuous and subcritical and critical nonlinear terms, respectively. We also note the recent contribution [2] where Ambrosio studied multiplicity and concentration of solutions for a fractional Kirchhoff equation with magnetic field and critical growth.

Motivated by [11, 13], in the present paper, our main goal is to study multiplicity and concentration of nontrivial solutions for problem (1.1) only when $f$ is continuous. Comparing with the result in [13], due to the presence of the nonlocal term, it is not clear to show the weak convergence of a bounded (PS) sequence of problem (1.1) is a solution of problem (1.1). Moreover, as we will see later, due to the presence of the magnetic field $A(x)$, problem (1.1) cannot be changed into a pure real-valued problem, and hence we should deal with a complex-valued problem directly, which causes several new difficulties in employing the methods in dealing with our problem. Our problem is more complicated than the pattern without magnetic field and we need additional technical estimates.

Throughout the paper, we make the following assumptions on the potential $V$:

(V1) There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$.

(V2) There exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$
V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial \Lambda} V(x).
$$

Observe that

$$
M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.
$$

Moreover, let the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying the following conditions:

(f1) $f(t) = 0$ if $t \leq 0$, and $\lim_{t \to 0^+} \frac{f(t)}{t} = 0$.

(f2) There exists $q \in (4, 6)$ such that

$$
\lim_{t \to +\infty} \frac{f(t)}{t^{\frac{q}{2}}} = 0.
$$
(f3) There is a positive constant $\theta > 4$ such that

$$0 < \frac{\theta}{2} F(t) \leq tf(t) \quad \text{for all } t > 0,$$

where $F(t) = \int_0^t f(s) \, ds$.

(f4) $\frac{f(t)}{t}$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following theorem.

**Theorem 1.1.** Assume that $V$ satisfies (V1), (V2) and $f$ satisfies (f1)–(f4). Then, for any $\delta > 0$ such that

$$M_\delta := \{ x \in \mathbb{R}^3 : \text{dist}(x, M) < \delta \} \subset \Lambda,$$

there exists $\epsilon_\delta > 0$ such that, for any $0 < \epsilon < \epsilon_\delta$, problem (1.1) has at least $\text{cat}_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0^+$ as $n \to +\infty$, if we denote by $u_{\epsilon_n}$ one of these solutions of problem (1.1) for $\epsilon = \epsilon_n$ and if $\eta_{\epsilon_n} \in \mathbb{R}^3$ is the global maximum point of $|u_{\epsilon_n}|$, then

$$\lim_{\epsilon_n \to 0^+} V(\eta_{\epsilon_n}) = V_0.$$ 

The paper is organized as follows. In Section 2, we introduce the functional setting and give some preliminaries. In Section 3, we study the modified problem and prove the Palais–Smale condition for the modified functional, and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the autonomous problem associated. It allows us to show that the modified problem has multiple solutions. Finally, in Section 5, we give the proof of Theorem 1.1.

**Notation.**

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line.
- $B_R(y)$ denotes the open ball centered at $y \in \mathbb{R}^3$ with radius $R > 0$, and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^3$.
- $\| \cdot \|_q$ and $\| \cdot \|_{L^q(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^3, \mathbb{R}), L^q(\mathbb{R}^3, \mathbb{R})$ and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^3$.

## 2 Abstract Setting

In this section, we introduce the function spaces and some useful preliminary remarks, which will be useful for our arguments.

For $u : \mathbb{R}^3 \to \mathbb{C}$, we set

$$\nabla_A u := \left( \frac{V}{I - A} \right) u.$$ 

Consider the function spaces

$$D^1_A(\mathbb{R}^3, \mathbb{C}) := \{ u \in L^6(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R}) \}$$

and

$$H^1_A(\mathbb{R}^3, \mathbb{C}) := \{ u \in D^1_A(\mathbb{R}^3, \mathbb{C}) : u \in L^2(\mathbb{R}^3, \mathbb{C}) \}.$$ 

The space $H^1_A(\mathbb{R}^3, \mathbb{C})$ is a Hilbert space endowed with the scalar product

$$\langle u, v \rangle := \text{Re} \int_{\mathbb{R}^3} (\nabla_A u \overline{\nabla_A v} + u\overline{v}) \, dx \quad \text{for any } u, v \in H^1_A(\mathbb{R}^3, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. We denote by $\|u\|_A$ the norm induced by this inner product, and $[u]_A^2 := \int_{\mathbb{R}^3} |\nabla_A u|^2 \, dx$. 

On $H^1_A(\mathbb{R}^3, \mathbb{C})$ we will frequently use the following diamagnetic inequality (see, e.g., [17, Theorem 7.21]):
\[
|\nabla_A u(x)| \geq |\nabla u(x)| \quad \text{for all} \quad u \in H^1_A(\mathbb{R}^3, \mathbb{C}).
\tag{2.1}
\]

Moreover, making a simple change of variables, since
\[
\Delta_A = \epsilon^2 \Delta_{A/\epsilon} \quad \text{and} \quad [u]_{A/\epsilon}^2 = \frac{1}{\epsilon} [u]_{A/\epsilon}^2,
\]
we can see that problem (1.1) is equivalent to
\[
- (a + b[u]^2_{A/\epsilon}) \Delta_{A/\epsilon} u + V_{\epsilon}(x) u = f(|u|^2) u \quad \text{in} \quad \mathbb{R}^3,
\tag{2.2}
\]
where $A_{\epsilon}(x) = A(\epsilon x)$ and $V_{\epsilon}(x) = V(\epsilon x)$.

Let $H_{\epsilon}$ be the Hilbert space obtained as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the scalar product
\[
\langle u, v \rangle_{\epsilon} := \text{Re} \int_{\mathbb{R}^3} (\nabla_A u \overline{\nabla_A \overline{v}} + V_{\epsilon}(x) u \overline{v}) \, dx
\]
and let $\| \cdot \|_{\epsilon}$ denote the norm induced by this inner product.

The diamagnetic inequality (2.1) implies that, if $u \in H^1_A(\mathbb{R}^3, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$ and $\|u\| \leq C \|u\|_{\epsilon}$.

Therefore, the embedding $H_{\epsilon} \hookrightarrow L^r(\mathbb{R}^3, \mathbb{C})$ is continuous for $2 \leq r \leq 6$ and the embedding $H_{\epsilon} \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{C})$ is compact for $1 \leq r < 6$.

### 3 The Modified Problem

As in [6], to study problem (1.1), or equivalently (2.2), we modify suitably the nonlinearity $f$ so that, for $\epsilon > 0$ small enough, the solutions of such modified problem are also solutions of the original one. More precisely, we choose $K > 2$. By (f4), there exists a unique number $a > 0$ verifying $K(a_0) = V_0$, where $V_0$ is given in (V1). Hence we consider the function
\[
\tilde{f}(t) := \begin{cases} 
 f(t), & t \leq a_0, \\
 V_0 - K t, & t > a_0.
\end{cases}
\]

Now we introduce the penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$:
\[
g(x, t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x)) \tilde{f}(t),
\]
where $\chi_\Lambda$ is the characteristic function on $\Lambda$. Set $G(x, t) := \int_0^t g(x, s) \, ds$.

In view of (f1)–(f4), we have that $g$ is a Carathéodory function satisfying the following properties:
\begin{enumerate}
\item[(g1)] $g(x, t) = 0$ for each $t \leq 0$.
\item[(g2)] $\lim_{t \to 0^+} \frac{g(x, t)}{t^2} = 0$ uniformly in $x \in \mathbb{R}^3$, and there exists $q \in (4, 6)$ such that
\[
\lim_{t \to +\infty} \frac{g(x, t)}{t^{q/2}} = 0 \quad \text{uniformly in} \quad x \in \mathbb{R}^3.
\]
\item[(g3)] $g(x, t) \leq f(t)$ for all $t \geq 0$ and uniformly in $x \in \mathbb{R}^3$.
\item[(g4)] $0 < \theta G(x, t) \leq 2g(x, t) t$ for each $x \in \Lambda$, $t > 0$.
\item[(g5)] $0 < G(x, t) \leq g(x, t) t \leq V_0 t / K$ for each $x \in \Lambda^c$, $t > 0$.
\item[(g6)] For each $x \in \Lambda$, the function $t \mapsto \frac{g(x, t)}{t}$ is strictly increasing in $t \in (0, +\infty)$, and for each $x \in \Lambda^c$ the function $t \mapsto \frac{g(x, t)}{t}$ is strictly increasing in $(0, a_0)$.
\end{enumerate}

Next, we consider the modified problem
\[
- (a + b[u]^2_{A/\epsilon}) \Delta_{A/\epsilon} u + V_{\epsilon}(x) u = g(\epsilon x, |u|^2) u \quad \text{in} \quad \mathbb{R}^3.
\tag{3.1}
\]

Note that, if $u$ is a solution of problem (3.1) with
\[
|u(x)|^2 \leq a_0 \quad \text{for all} \quad x \in \Lambda^c,
\]
then $u$ is a solution of problem (2.2).
The functional associated to (3.1) is
\[ J_\epsilon(u) := \frac{a}{2} \| u \|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\epsilon(x) |u|^2 \, dx + \frac{b}{4} [u]_{L^4}^4 - \frac{1}{2} \int_{\mathbb{R}^3} G(\epsilon x, |u|^2) \, dx \]
defined in \( H_\epsilon \). It is standard to prove that \( J_\epsilon \in C^1(H_\epsilon, \mathbb{R}) \) and its critical points are the weak solutions of the modified problem (3.1).

We denote by \( N_\epsilon \) the Nehari manifold of \( J_\epsilon \), that is,
\[ N_\epsilon := \{ u \in H_\epsilon \setminus \{ 0 \} : J_\epsilon'(u)[u] = 0 \}, \]
and define the number \( c_\epsilon \) by
\[ c_\epsilon = \inf_{u \in N_\epsilon} J_\epsilon(u). \]
Let \( H_\epsilon^r \) be the open subset \( H_\epsilon \) given by
\[ H_\epsilon^r = \{ u \in H_\epsilon : |\text{supp}(u) \cap \Lambda_\epsilon| > 0 \}, \]
and \( S_\epsilon^+ = S_\epsilon \cap H_\epsilon^r \), where \( S_\epsilon \) is the unit sphere of \( H_\epsilon \). Note that \( S_\epsilon^+ \) is a non-complete \( C^{1,1} \)-manifold of codimension 1, modeled on \( H_\epsilon \) and contained in \( H_\epsilon^r \). Therefore, \( H_\epsilon = T_u S_\epsilon^+ \oplus \mathbb{R} u \) for each \( u \in T_u S_\epsilon^+ \), where \( T_u S_\epsilon^+ = \{ v \in H_\epsilon : \langle u, v \rangle_\epsilon = 0 \}. \)

Now we show that the functional \( J_\epsilon \) satisfies the mountain pass geometry.

**Lemma 3.1.** For any fixed \( \epsilon > 0 \), the functional \( J_\epsilon \) satisfies the following properties:
(i) There exist \( \beta, \rho > 0 \) such that \( J_\epsilon(u) \geq \beta \) if \( \| u \|_\epsilon = \rho. \)
(ii) There exists \( e \in H_\epsilon \) with \( \| e \|_\epsilon > \rho \) such that \( J_\epsilon(e) < 0. \)

**Proof.** (i) By (g2), (g4) and (g5), for any \( \xi > 0 \) small, there exists \( C_\xi > 0 \) such that
\[ G(\epsilon x, |u|^2) \leq \xi |u|^4 + C_\xi |u|^q \quad \text{for all} \quad x \in \mathbb{R}^3. \]
By the Sobolev embedding theorem, it follows that
\[ J_\epsilon(u) \geq \frac{a}{2} \| u \|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\epsilon(x) |u|^2 \, dx + \frac{b}{4} [u]_{L^4}^4 - \frac{\xi}{2} \int_{\mathbb{R}^3} |u|^4 \, dx - C_\xi \frac{\| u \|_\epsilon^q}{2} \]
\[ \geq \frac{1}{2} \| u \|_\epsilon^2 - C_1 \xi \| u \|_\epsilon^2 - C_2 C_\xi \| u \|_\epsilon^q. \]
Hence we can choose some \( \beta, \rho > 0 \) such that \( J_\epsilon(u) \geq \beta \) if \( \| u \|_\epsilon = \rho \) since \( q > 4. \)
(ii) For each \( u \in H_\epsilon^r \) and \( t > 0 \), by the definition of \( g \) and (f3), one has
\[ J_\epsilon(tu) \leq \frac{t^2}{2} \| u \|_\epsilon^2 + \frac{bt^4}{4} [u]_{L^4}^4 - \frac{1}{4} \int_{\mathbb{R}^3} G(\epsilon x, t^2 |u|^2) \, dx \]
\[ \leq \frac{t^2}{2} \| u \|_\epsilon^2 + \frac{bt^4}{4} [u]_{L^4}^4 - C_1 t^\theta \int_{\Lambda_\epsilon} |u|^\theta \, dx + C_2 |\text{supp}(u) \cap \Lambda_\epsilon|. \]
Since \( \theta > 4 \), we can get the conclusion. \( \square \)

Since \( f \) is only continuous, the next results are very important because they allow us to overcome the non-differentiability of \( N_\epsilon \) and the incompleteness of \( S_\epsilon^+ \).

**Lemma 3.2.** Assume that (V1)–(V2) and (f1)–(f4) are satisfied. Then the following properties hold:
(A1) For any \( u \in H_\epsilon^r \), let \( g_u : \mathbb{R}^+ \to \mathbb{R} \) be given by \( g_u(t) = J_\epsilon(tu) \). Then there exists a unique \( t_u > 0 \) such that \( g_u'(t) > 0 \) in \((0, t_u)\) and \( g_u'(t) < 0 \) in \((t_u, \infty). \)
(A2) There is some \( \tau > 0 \) independent of \( u \) such that \( t_u \geq \tau \) for all \( u \in S_\epsilon^+ \). Moreover, for each compact \( \mathcal{W} \subset S_\epsilon^+ \) there is a constant \( C_\mathcal{W} \) such that \( t_u \leq C_\mathcal{W} \) for all \( u \in \mathcal{W} \).
(A3) The map \( \tilde{m}_e : H^1_0 \to N_e \) given by \( \tilde{m}_e(u) = t_u u \) is continuous, and \( m_e = \tilde{m}_e|_{S^e_\delta} \) is a homeomorphism between \( S^e_\delta \) and \( N_e \). Moreover, \( m_e^{-1}(u) = \frac{u}{\|u\|} \).

(A4) If there is a sequence \( \{u_n\} \subset S^e_\delta \) such that \( \text{dist}(u_n, \partial S^e_\delta) \to 0 \), then \( \|m_e(u_n)\|_e \to \infty \) and \( J_e(m_e(u_n)) \to \infty \).

Proof. (A1) As in the proof of Lemma 3.1, we have \( g_u(0) = 0 \), \( g_u(t) > 0 \) for \( t > 0 \) small, and \( g_u(t) < 0 \) for \( t > 0 \) large. Therefore, \( \max_{t \geq 0} g_u(t) \) is achieved at a global maximum point \( t = t_u \) verifying \( g_u(t_u) = 0 \) and \( t_u u \in N_e \). Now, we show that \( t_u \) is unique. Arguing by contradiction, suppose that there exist \( t_1 > t_2 > 0 \) such that \( g_u(t_1) = g_u(t_2) = 0 \). Then, for \( i = 1, 2 \),

\[
t_i a[u]_{A_\delta} + t_i \int_{\mathbb{R}^3} V_e(x)|u|^2 \, dx + t_i \beta_i b[u]_{A_\delta}^4 = \int_{\mathbb{R}^3} g(ex, t_i^2|u|^2) t_i |u|^2 \, dx.
\]

Hence,

\[
a[u]_{A_\delta} + \int_{\mathbb{R}^3} V_e(x)|u|^2 \, dx + \beta_i b[u]_{A_\delta}^4 = \int_{\mathbb{R}^3} g(ex, t_i^2|u|^2) |u|^2 \, dx,
\]

which implies that

\[
\left( \frac{1}{t_1^2} - \frac{1}{t_2^2} \right) a[u]_{A_\delta} + \int_{\mathbb{R}^3} V_e(x)|u|^2 \, dx \\
= \int_{\mathbb{R}^3} \left( \frac{g(ex, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(ex, t_2^2|u|^2)}{t_2^2|u|^2} \right) |u|^2 \, dx \\
\geq \int_{A_\delta \cap \{t_1^2|u|^2 \leq t_2^2|u|^2 \}} \left( \frac{g(ex, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(ex, t_2^2|u|^2)}{t_2^2|u|^2} \right) |u|^2 \, dx
\]

\[
+ \int_{A_\delta \cap \{t_2^2|u|^2 \leq t_1^2|u|^2 \}} \left( \frac{g(ex, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(ex, t_2^2|u|^2)}{t_2^2|u|^2} \right) |u|^2 \, dx
\]

\[
\geq \int_{A_\delta \cap \{t_1^2|u|^2 \leq t_2^2|u|^2 \}} \left( \frac{V_0}{K} \frac{1}{t_1^2|u|^2} - \frac{f(t_2^2|u|^2)}{t_2^2|u|^2} \right) |u|^2 \, dx + \frac{1}{K} \left( \frac{1}{t_1^2} - \frac{1}{t_2^2} \right) \int_{A_\delta \cap \{t_2^2|u|^2 \leq t_1^2|u|^2 \}} V_0 |u|^2 \, dx
\]

Since \( t_1 > t_2 > 0 \), we have

\[
\left( a[u]_{A_\delta} + \int_{\mathbb{R}^3} V_e(x)|u|^2 \, dx \right) \\
\leq \frac{t_1^2}{t_2^2 - t_1^2} \int_{A_\delta \cap \{t_1^2|u|^2 \leq t_2^2|u|^2 \}} \left( \frac{V_0}{K} \frac{1}{t_1^2|u|^2} - \frac{f(t_2^2|u|^2)}{t_2^2|u|^2} \right) |u|^2 \, dx + \frac{1}{K} \int_{A_\delta \cap \{t_2^2|u|^2 \leq t_1^2|u|^2 \}} V_0 |u|^2 \, dx
\]

\[
\leq \frac{1}{K} \int_{A_\delta} V_0 |u|^2 \, dx
\]

\[
\leq \frac{1}{K} \|u\|_e^2,
\]

which is a contradiction. Therefore, \( \max_{t \geq 0} g_u(t) \) is achieved at a unique \( t = t_u \) so that \( g_u'(t_u) = 0 \) and \( t_u u \in N_e \).

(A2) For any \( u \in S^e_\delta \), we have

\[
t_u + t_u^3 b[u]_{A_\delta}^4 = \int_{\mathbb{R}^3} g(ex, t_u^2|u|^2) t_u |u|^2 \, dx.
\]

From (g2), Sobolev embeddings and since \( q > 4 \), we get

\[
t_u \leq \zeta_{t_u}^3 \int_{\mathbb{R}^3} |u|^6 \, dx + C \zeta_{t_u}^{q-1} \int_{\mathbb{R}^3} |u|^q \, dx \leq C_1 \zeta_{t_u}^3 + C_2 C \zeta_{t_u}^{q-1},
\]
which implies that $t_n \geq \tau$ for some $\tau > 0$. Suppose by contradiction that there is $\{u_n\} \subset W$ with $t_n := t_{u_n} \to \infty$. Since $W$ is compact, there exists $u \in W$ such that $u_n \to u$ in $H_c$. Moreover, using the proof of Lemma 3.1 (ii), we have that $J_c(t_n u_n) \to -\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_c$. From the definition of $g$ and by (g4), (g5) and $\theta > 4$, it follows that

$$J_c(v_n) = J_c(v_n) - \frac{1}{\theta} J_c'(v_n)[v_n]$$

$$\geq \left(1 - \frac{1}{\theta}\right) \|v_n\|_e^2 + \left(1 - \frac{1}{\theta}\right) b[v_n]A_c^2 + \int_{\Lambda_c} \left(\frac{1}{\theta} g(\epsilon x, |v_n|^2)|v_n|^2 - \frac{1}{2} G(\epsilon x, |v_n|^2)\right) dx$$

$$\geq \left(1 - \frac{1}{\theta}\right) \|v_n\|_e^2 - \frac{1}{\theta} \int_{\mathbb{R}^3} V(\epsilon x)|v_n|^2 dx$$

$$\geq \left(1 - \frac{1}{\theta}\right) \|v_n\|_e^2.$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_e = t_n$, we obtain

$$0 < \left(\frac{1}{\theta} - 1\right) t_n \leq \frac{J_c(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This proves (A2).

(A3) First of all, we note that $\tilde{m}_e, m_e$ and $m^{-1}_e$ are well defined. Indeed, by (A2), for each $u \in H^*_c$, there is a unique $\tilde{m}_e(u) \in \mathcal{N}_e$. On the other hand, if $u \in \mathcal{N}_e$, then $u \in H^*_c$. Otherwise, we have $|\text{supp}(u) \cap \Lambda_e| = 0$ and by (g5), we have

$$a[u]_{A_c}^2 + \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2 dx + b[u]_{A_c}^2 = \int_{\mathbb{R}^3} g(\epsilon x, |u|^2)|u|^2 dx$$

$$= \int_{\Lambda_c} g(\epsilon x, |u|^2)|u|^2 dx$$

$$\leq \frac{1}{\theta} \int_{\mathbb{R}^3} V(\epsilon x)|u|^2 dx$$

$$\leq \frac{1}{\theta} \|u\|_e^2,$$

which is impossible since $K > 2$ and $u \neq 0$. Therefore, $m_e^{-1}(u) = \frac{u}{\|u\|_e} \in S^*_c$ is well defined and continuous. From

$$m_e^{-1}(m_e(u)) = m_e^{-1}(t_n u) = \frac{t_n u}{\|t_n u\|_e} = u \quad \text{for all } u \in S^*_c,$$

we conclude that $m_e$ is a bijection.

Now we prove that $\tilde{m}_e : H^*_c \to \mathcal{N}_e$ is continuous. Let $\{u_n\} \subset H^*_c$ and $u \in H^*_c$ such that $u_n \to u$ in $H_c$. By (A2), there exists $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_e$, that is,

$$t_n^2 a[u_n]_{A_c}^2 + t_n^2 \int_{\mathbb{R}^3} V_\epsilon(x)|u_n|^2 dx + t_n^2 b[u_n]_{A_c}^2 = \int_{\mathbb{R}^3} g(\epsilon x, t_n^2)|u_n|^2 dx$$

for all $n \in \mathbb{N}$,

and passing to the limit as $n \to \infty$ in the last inequality, we obtain

$$t_0^2 a[u]_{A_c}^2 + t_0^2 \int_{\mathbb{R}^3} V_\epsilon(x)|u|^2 dx + t_0^2 b[u]_{A_c}^2 = \int_{\mathbb{R}^3} g(\epsilon x, t_0^2)|u|^2 dx,$$

which implies that $t_0 u \in \mathcal{N}_e$ and $t_0 = t_0$. This proves that $\tilde{m}_e(u_n) \to \tilde{m}_e(u)$ in $H^*_c$. Thus, $\tilde{m}_e$ and $m_e$ are continuous functions and (A3) is proved.

(A4) Let $\{u_n\} \subset S^*_c$ be a subsequence such that $\text{dist}(u_n, \partial S^*_c) \to 0$. Then, for each $v \in \partial S^*_c$ and $n \in \mathbb{N}$, we have $|u_n| = |u_n - v|$ a.e. in $\Lambda_c$. Therefore, by (V1), (V2) and the Sobolev embedding theorem, there exists
a constant $C_r > 0$ such that
\[
\|u_n\|_{L^r(A)} \leq \inf_{v \in \partial S^n_e} \|u_n - v\|_{L^r(A)}
\]
\[
\leq C_r \left( \inf_{v \in \partial S^n_e} \left( \int \left( |\nabla A_n u_n - v| + V_{\epsilon}(x)|u_n - v| \right)^2 \right)^{\frac{1}{2}} \right)
\]
\[
\leq C_r \text{dist}(u_n, \partial S^n_e)
\]
for all $n \in \mathbb{N}$ and $r \in [2, 6]$. By (g2), (g3) and (g5), for each $t > 0$, we have
\[
\int_{\mathbb{R}^n} G(\epsilon x, t^2|u_n|^2) \, dx \leq \int_{\Lambda_\epsilon} F(t^2|u_n|^2) \, dx + \frac{t^2}{K} \int_{\Lambda_\epsilon} V(\epsilon x)|u_n|^2 \, dx
\]
\[
\leq C_1 t^4 \int_{\Lambda_\epsilon} |u_n|^4 \, dx + C_2 t^4 \int_{\Lambda_\epsilon} |u_n|^4 \, dx + \frac{t^2}{K} \|u_n\|_\epsilon^2
\]
\[
\leq C_3 t^4 \text{dist}(u_n, \partial S^n_e)^4 + C_4 t^4 \text{dist}(u_n, \partial S^n_e)^4 + \frac{t^2}{K}.
\]
Therefore,
\[
\limsup_n \int_{\mathbb{R}^n} G(\epsilon x, t^2|u_n|^2) \, dx \leq \frac{t^2}{K} \text{ for all } t > 0.
\]
On the other hand, from the definition of $m_\epsilon$ and the last inequality, for all $t > 0$, one has
\[
\liminf_n \inf_{u \in H^+ \subset H^2} J_\epsilon(m_\epsilon(u_n)) \geq \liminf_n \inf_{u \in H^+ \subset H^2} J_\epsilon(tu_n)
\]
\[
\geq \liminf_n \frac{t^2}{2} \|u_n\|^2 - \frac{t^2}{K}
\]
\[
= \frac{K - 2}{2K} t^2.
\]
This implies that
\[
\liminf_n \frac{1}{2} \|m_\epsilon(u_n)\|^2 \geq \frac{K - 2}{2K} t^2 \text{ for all } t > 0.
\]
From the arbitrariness of $t > 0$, it is easy to see that $\|m_\epsilon(u_n)\|_\epsilon \to \infty$ and $J_\epsilon(m_\epsilon(u_n)) \to \infty$ as $n \to \infty$. This completes the proof of Lemma 3.2.

Now we define the function
\[
\bar{\Psi}_\epsilon : \mathcal{C}^1(\mathcal{H}^+ \subset \mathcal{R}) \to \mathbb{R}
\]
by $\bar{\Psi}_\epsilon(u) = J_\epsilon(\bar{m}_\epsilon(u))$ and set $\Psi_\epsilon := (\bar{\Psi}_\epsilon)|_{S_\epsilon^n}$.

From Lemma 3.2, we have the following result.

**Lemma 3.3.** Assume that (V1)–(V2) and (f1)–(f4) are satisfied. Then the following assertions hold:

(B1) $\Psi_\epsilon \in C^1(H^+_\epsilon \subset \mathcal{R})$ and
\[
\bar{\Psi}_\epsilon'(u) v = \frac{\|\bar{m}_\epsilon(u)\|_\epsilon}{\|u\|_\epsilon} \frac{f_\epsilon'(\bar{m}_\epsilon(u))}{\|v\|_\epsilon} \quad \text{for all } u \in \mathcal{H}^+_\epsilon \text{ and all } v \in \mathcal{H}^\epsilon.
\]

(B2) $\Psi_\epsilon \in C^1(S^\epsilon \subset \mathcal{R})$ and
\[
\Psi_\epsilon'(u) v = \|m_\epsilon(u)\|_\epsilon f_\epsilon'(m_\epsilon(u)) \quad \text{for all } v \in T^\epsilon u \subset S^\epsilon.
\]

(B3) If $[u_n]$ is a (PS)$_\epsilon$ sequence of $\Psi_\epsilon$, then $[m_\epsilon(u_n)]$ is a (PS)$_\epsilon$ sequence of $J_\epsilon$. If $[u_n] \subset N_\epsilon$ is a bounded (PS)$_\epsilon$ sequence of $J_\epsilon$, then $[m_\epsilon^{-1}(u_n)]$ is a (PS)$_\epsilon$ sequence of $\Psi_\epsilon$.

(B4) $u$ is a critical point of $\Psi_\epsilon$ if and only if $m_\epsilon(u)$ is a critical point of $J_\epsilon$. Moreover, the corresponding critical values coincide and
\[
\inf_{S^\epsilon_\epsilon} \Psi_\epsilon = \inf_{N_\epsilon} J_\epsilon.
\]
As in [21], we have the following variational characterization of the infimum of $J_c$ over $N_c$:

$$c_c = \inf_{u \in N_c} J_c(u) = \inf_{u \in H^1_0} \sup_{t > 0} J_c(tu) = \inf_{u \in S^1_c} \sup_{t > 0} J_c(tu).$$

**Lemma 3.4.** Let $c > 0$ and let $\{u_n\}$ be a $(PS)_c$ sequence for $J_c$. Then $\{u_n\}$ is bounded in $H^1_0$.

**Proof.** Assume that $\{u_n\} \subset H^1_0$ is a $(PS)_c$ sequence for $J_c$, that is, $J_c(u_n) \to c$ and $J'_c(u_n) \to 0$. By using (g4), (g5) and $\theta > 4$, we have

$$c + o_n(1) + o_n(1)\|u_n\|_c \geq J_c(u_n) - \frac{1}{\theta} J'_c(u_n)[u_n] \geq \frac{1}{2} \|u_n\|_c^2 + \frac{1}{4} b[u_n]_A^2 + \frac{1}{2} \int_{\mathbb{R}^3} g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\epsilon x, |u_n|^2) \ dx$$

$$\geq \frac{1}{2} \|u_n\|_c^2 + \frac{1}{2} \int_{\mathbb{R}^3} G(\epsilon x, |u_n|^2) \ dx$$

$$\geq \frac{1}{2} \|u_n\|_c^2 + \frac{1}{2K} \int_{\mathbb{R}^3} V(\epsilon x)|u_n|^2 \ dx$$

$$\geq \frac{1}{2} \|u_n\|_c^2 + \frac{1}{2K} \|u_n\|_c^2.$$

Since $K > 2$, from the above inequalities we obtain that $\{u_n\}$ is bounded in $H^1_0$. \hfill \Box

The following result is important to prove the $(PS)_c$ condition for the functional $J_c$.

**Lemma 3.5.** The functional $J_c$ satisfies the $(PS)_c$ condition at any level $c > 0$.

**Proof.** Let $(u_n) \subset H^1_0$ be a $(PS)_c$ sequence for $J_c$. By Lemma 3.4, $(u_n)$ is bounded in $H^1_0$. Thus, up to a subsequence, $u_n \rightharpoonup u$ in $H^1_0$ and $u_n \to u$ in $L^r(\mathbb{R}^3, C)$ for all $1 \leq r < 6$ as $n \to +\infty$. Moreover, $J'_c(u) = 0$ and

$$a[u]_A^2 + \int_{\mathbb{R}^3} V_c(x)|u|^2 \ dx + b[u]_A^2 = \int_{\mathbb{R}^3} g(\epsilon x, |u|^2)|u|^2 \ dx.$$

For the fixed $\epsilon > 0$, let $R > 0$ be such that $\Lambda_{\epsilon} \subset B_{R/2}(0)$. We show that for any given $\zeta > 0$, for $R$ large enough,

$$\limsup_{n} \int_{B_R(0)} (|\nabla u_n|^2 + V_c(x)|u_n|^2) \ dx \leq \zeta.$$  \hfill (3.2)

Let $\phi_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \ for \ x \in B_{R/2}(0), \quad \phi_R = 1 \ for \ x \in B_R^c(0), \quad 0 \leq \phi_R \leq 1, \quad |\nabla \phi_R| \leq \frac{C}{R},$$

where $C > 0$ is a constant independent of $R$. Since the sequence $(\phi_R u_n)$ is bounded in $H^1_0$, we have

$$J'_c(u_n)[\phi_R u_n] = o_n(1),$$

that is

$$a \text{Re} \int_{\mathbb{R}^3} \nabla A_u u_n \nabla A_u (\phi_R u_n) \ dx + \int_{\mathbb{R}^3} V_c(x)|u_n|^2 \phi_R \ dx + b[u_n]_A^2 \text{Re} \int_{\mathbb{R}^3} \nabla A_u u_n \nabla A_u (\phi_R u_n) \ dx$$

$$= \int_{\mathbb{R}^3} g(\epsilon x, |u_n|^2)|u_n|^2 \phi_R \ dx + o_n(1).$$

Since

$$\nabla A_u (u_n \phi_R) = i u_n \nabla \phi_R + \phi_R \nabla A_u u_n,$$
By the definition of $\phi_R$, the Hölder inequality and the boundedness of $(u_n)$ in $H_\varepsilon$, we obtain
\[\left(1 - \frac{1}{R}\right) \int_{\mathbb{R}^3} (a|\nabla A_n u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R \, dx \leq \frac{C}{R} \|u_n\|_2 \|\nabla A_n u_n\|_2 + o_n(1) \leq \frac{C_1}{R} + o_n(1),\]
and so (3.2) holds.

Now, we prove that for any $R > 0$ the following limit holds:
\[
\limsup_{n \to \infty} \int_{B_R(0)} (|\nabla A_n u_n|^2 + V_\varepsilon(x)|u_n|^2) \, dx = \int_{B_R(0)} (|\nabla A_n u|^2 + V_\varepsilon(x)|u|^2) \, dx. \tag{3.3}
\]
Let $\phi_\rho \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that
\[\phi_\rho = 1 \text{ for } x \in B_\rho(0), \quad \phi_\rho = 0 \text{ for } x \in B_{2\rho}(0), \quad 0 \leq \phi_\rho \leq 1, \quad |\nabla \phi_\rho| \leq \frac{C_1}{\rho},\]
where $C_1 > 0$ is a constant independent of $\rho$. Let
\[P_n(x) = M(u_n)|\nabla A_n u_n - \nabla A_n u|^2 + V_\varepsilon(x)|u_n - u|^2,\]
where
\[M(u_n) = a + b \int_{\mathbb{R}^3} |\nabla A_n u_n|^2 \, dx.\]
For the fixed $R > 0$, choosing $\rho > R > 0$, we have
\[
\int_{B_R} P_n(x) \, dx \leq \int_{B_R} P_n(x) \phi_\rho(x) \, dx = M(u_n) \int_{\mathbb{R}^3} |\nabla A_n u_n - \nabla A_n u|^2 \phi_\rho(x) \, dx + \int_{\mathbb{R}^3} V_\varepsilon(x)|u_n - u|^2 \phi_\rho(x) \, dx = J_{1,\rho} - J_{2,\rho} + J_{3,\rho} + J_{4,\rho}, \tag{3.4}
\]
where
\[
J_{1,\rho} = M(u_n) \int_{\mathbb{R}^3} |\nabla A_n u_n|^2 \phi_\rho(x) \, dx + \int_{\mathbb{R}^3} V_\varepsilon(x)|u_n|^2 \phi_\rho(x) \, dx - \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2)|u_n|^2 \phi_\rho(x) \, dx,
\]
\[
J_{2,\rho} = M(u_n) \text{ Re} \int_{\mathbb{R}^3} \nabla A_n u_n \overline{\nabla A_n u_n} \overline{\phi_\rho(x)} \, dx + \text{ Re} \int_{\mathbb{R}^3} V_\varepsilon(x) u_n \overline{u_n} \overline{\phi_\rho(x)} \, dx - \text{ Re} \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2) u_n \overline{u_n} \phi_\rho(x) \, dx,
\]
\[
J_{3,\rho} = -M(u_n) \text{ Re} \int_{\mathbb{R}^3} (\nabla A_n u_n - \nabla A_n u) \overline{\nabla A_n u_n} \phi_\rho(x) \, dx + \text{ Re} \int_{\mathbb{R}^3} V_\varepsilon(x) (u_n - u) \overline{u_n} \phi_\rho(x) \, dx,
\]
\[
J_{4,\rho} = \text{ Re} \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2) u_n (u_n - u) \phi_\rho(x) \, dx.
\]
It is easy to see that
\[
J_{1,\rho} = J'_{\varepsilon}(u_n)[\phi_\rho u_n] - M(u_n) \text{ Re} \int_{\mathbb{R}^3} \overline{u_n} \nabla A_n u_n \nabla \phi_\rho \, dx
\]
and

\[ J_{n,\rho}^2 = J_{n,\rho}^1(u_n)[\phi_\rho u] - M(u_n) \text{Re} \int_{\mathbb{R}^3} i\nabla A_n u_n \nabla \phi_\rho \, dx. \]

Then

\[ \lim_{\rho \to \infty} \limsup_{n \to \infty} |J_{n,\rho}^1| = 0, \quad \lim_{\rho \to \infty} \limsup_{n \to \infty} |J_{n,\rho}^2| = 0. \]

On the other hand, since the sequence \((u_n)\) is bounded in \(H_\epsilon\), we assume that

\[ \int_{\mathbb{R}^3} |\nabla A_n u_n|^2 \, dx \to I^2. \]

Then

\[ J_{n,\rho}^3 = -(a + b\epsilon^2) \text{Re} \int_{\mathbb{R}^3} (\nabla A_n u_n - \nabla A_n u) \nabla A_n (u \phi_\rho(x)) \, dx - \text{Re} \int_{\mathbb{R}^3} V_\epsilon(x)(u_n - u)(u \phi_\rho(x)) \, dx \]

\[ + (a + b\epsilon^2) \text{Re} \int_{\mathbb{R}^3} (\nabla A_n u_n - \nabla A_n u) \nabla \phi_\rho \, dx + o_n(1) \]

\[ = -(a + b\epsilon^2) \langle u_n - u, u \phi_\rho(x) \rangle + (a + b\epsilon^2) \text{Re} \int_{\mathbb{R}^3} (\nabla A_n u_n - \nabla A_n u) \nabla \phi_\rho \, dx + o_n(1), \]

and thus

\[ \lim_{\rho \to \infty} \limsup_{n \to \infty} |J_{n,\rho}^3| = 0. \]

Now we prove that

\[ \lim_{\rho \to \infty} \limsup_{n \to \infty} |J_{n,\rho}^4| = 0. \] (3.5)

It is easy to see that

\[ J_{n,\rho}^4 \leq \int_{(\mathbb{R}^3 \setminus A_n) \cap B_{1\rho}(0)} |g(ex, |u_n|^2)u_n(u_n - u)| \, dx + \int_{A_n \cap B_{2\rho}(0)} |g(ex, |u_n|^2)u_n(u_n - u)| \, dx. \]

Using the Sobolev compact embedding \(H_\epsilon \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^3, \mathbb{C})\) for \(1 \leq r < 6\), (g5), (f1) and (f2) imply that

\[ \int_{(\mathbb{R}^3 \setminus A_n) \cap B_{1\rho}(0)} |g(ex, |u_n|^2)u_n(u_n - u)| \, dx \to 0 \quad \text{as} \quad n \to \infty \]

and

\[ \int_{A_n \cap B_{2\rho}(0)} |g(ex, |u_n|^2)u_n(u_n - u)| \, dx \to 0 \quad \text{as} \quad n \to \infty. \]

Thus, (3.5) holds. Moreover, by (3.4), it follows that

\[ 0 \leq \liminf_n \int_{B_\rho} P_n(x) \, dx \leq \limsup_n |J_{n,\rho}^1| + |J_{n,\rho}^2| + |J_{n,\rho}^3| + |J_{n,\rho}^4| = 0. \]

Then

\[ \limsup_n \int_{B_\rho} P_n(x) \, dx = 0. \]

Thus, (3.3) holds. Finally, from (3.2) and (3.3), we have

\[ \|u\|_\epsilon^2 \leq \liminf_n \|u_n\|_\epsilon^2 \]

\[ \leq \limsup_n \|u_n\|_\epsilon^2 \]

\[ \leq \limsup_n \left\{ \int_{B_\rho(0)} (a|\nabla A_n u_n|^2 + V_\epsilon(x)|u_n|^2) \, dx + \int_{B_\rho(0)} (a|\nabla A_n u_n|^2 + V_\epsilon(x)|u_n|^2) \, dx \right\} \]

\[ \leq \int_{B_\rho(0)} (a|\nabla A_\epsilon u|^2 + V_\epsilon(x)|u|^2) \, dx + \zeta. \]
Passing to the limit as $\zeta \to 0$, we have $R \to \infty$, which implies that
\[ \|u\|_\infty^2 \leq \lim \inf_n \|u_n\|_\infty^2 \leq \lim \sup_n \|u_n\|_\infty^2 \leq \|u\|_\infty^2. \]

Then $u_n \to u$ in $H_\epsilon$, and we complete the proof of this theorem. \hfill \Box

Since $f$ is only assumed to be continuous, the following result is required for the multiplicity result in the next section.

**Corollary 3.6.** The functional $\Psi_\epsilon$ satisfies the (PS)$_c$ condition on $S_\epsilon^*$ at any level $c > 0$.

**Proof.** Let $\{u_n\} \subset S_\epsilon^*$ be a (PS)$_c$ sequence for $\Psi_\epsilon$. Then $\Psi_\epsilon(u_n) \to c$ and $\|\Psi'_\epsilon(u_n)\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_\infty}S_\epsilon^*)^*$. By Lemma 3.3(B3), we know that $\{m_\epsilon(u_n)\}$ is a (PS)$_c$ sequence for $I_\epsilon$ in $H_\epsilon$. From Lemma 3.5, we know that there exists a $u \in S_\epsilon^*$ such that, up to a subsequence, $m_\epsilon(u_n) \to m_\epsilon(u)$ in $H_\epsilon$. By Lemma 3.2(A3), we obtain
\[ u_n \to u \quad \text{in} \quad S_\epsilon^*, \]
and the proof is complete. \hfill \Box

**Proposition 3.7.** Assume that (V1)–(V2) and (f1)–(f4) hold. Then problem (3.1) has a ground state solution for any $\epsilon > 0$.

**Proof.** From Lemma 3.1 and Lemma 3.5, we can obtain the existence of a ground state $u \in H_\epsilon$ for problem (3.1). \hfill \Box

## 4 Multiplication Solutions for the Modified Problem

### 4.1 The Autonomous Problem

For our scope, we also need to study the following limit problem:
\[ -(a + b|u|^2)\Delta u + V_0 u = f(|u|^2)u, \quad u : \mathbb{R}^3 \to \mathbb{R}, \quad (4.1) \]
whose associated $C^1$-functional, defined on $H^1(\mathbb{R}^3, \mathbb{R})$, is
\[ I_0(u) := \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + V_0 u^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 - \frac{1}{2} \int_{\mathbb{R}^3} F(u^2) \, dx. \]

Let
\[ N_0 := \{ u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} : I_0'(u)[u] = 0 \} \]
and
\[ c_{V_0} := \inf_{u \in N_0} I_0(u). \]

Let $S_0$ be the unit sphere of $H_0 := H^1(\mathbb{R}^3, \mathbb{R})$ and let it be a complete and smooth manifold of codimension 1. Therefore, $H_0 = T_u S_0 \bigoplus \mathbb{R} u$ for each $u \in T_u S_0$, where $T_u S_0 = \{ v \in H_0 : \langle u, v \rangle_0 = 0 \}$.

**Lemma 4.1.** Let $V_0$ be given in (V1) and suppose that (f1)–(f4) are satisfied. Then the following properties hold:
(a1) For any $u \in H_0 \setminus \{0\}$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_0(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in $(t_u, \infty)$.
(a2) There is a $\tau > 0$ independent of $u$ such that $t_u > \tau$ for all $u \in S_0$. Moreover, for each compact $\mathcal{W} \subset S_0$ there exists a $t_u$ such that $t_u \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.
(a3) The map $\tilde{m} : H_0 \setminus \{0\} \to N_0$ given by $\tilde{m}(u) = t_u u$ is continuous, and $m_0 = \tilde{m}|S_0$ is a homeomorphism between $S_0$ and $N_0$. Moreover, $m^{-1}(u) = \frac{u}{|m_0(u)|}$.
(a4) If there is a sequence $\{u_n\} \subset S_0$ such that $\text{dist}(u_n, \partial S_0) \to 0$, then $\|m(u_n)\|_0 \to \infty$ and $I_0(m(u)) \to \infty$ as $n \to \infty$.  

...
Lemma 4.2. Let $V_0$ be given in (V1) and suppose that ($f_1$)–($f_4$) are satisfied. Then the following assertions hold:

(b1) $\Psi_0 \in C^1(H_0 \setminus \{0\}, \mathbb{R})$ and

\[
\Psi_0'(u)v = \frac{\|\tilde{m}(u)\|_0 I_0'(\tilde{m}(u))[v]}{\|u\|_0} \quad \text{for all } u \in H_0 \setminus \{0\} \text{ and all } v \in H_0.
\]

(b2) $\Psi_0 \in C^1(S_0, \mathbb{R})$ and

\[
\Psi_0'(u)v = \|m(u)\|_0 I_0'(\tilde{m}(u))[v] \quad \text{for all } v \in T_n S_0.
\]

(b3) If $\{u_n\}$ is a $(PS)_c$ sequence of $\Psi_0$, then $\{m(u_n)\}$ is a $(PS)_c$ sequence of $I_0$. If $\{u_n\} \subset N_0$ is a bounded $(PS)_c$ sequence of $I_0$, then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence of $\Psi_0$.

(b4) We have that $u$ is a critical point of $\Psi_0$ if and only if $m(u)$ is a critical point of $I_0$. Moreover, the corresponding critical values coincide and

\[
\inf_{S_0} \Psi_0 = \inf_{N_0} I_0.
\]

Similarly to the previous argument, we have the following variational characterization of the infimum of $I_0$ over $N_0$:

\[
c_{V_0} = \inf_{u \in N_0} I_0(u) = \sup_{u \in H_0 \setminus \{0\}} \inf_{t > 0} I_0(tu) = \sup_{u \in S_0} \inf_{t > 0} I_0(tu).
\]

The next result is useful in later arguments.

Lemma 4.3. Let $\{u_n\} \subset H_0$ be a $(PS)_c$ sequence for $I_0$ such that $u_n \rightarrow 0$. Then one of the following alternatives occurs:

(i) $u_n \rightarrow 0$ in $H_0$ as $n \rightarrow +\infty$.

(ii) There are a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

\[
\liminf_n \int_{B_R(y_n)} |u_n|^2 \geq \beta.
\]

Proof. Assume that (ii) does not hold. Then, for every $R > 0$, we have

\[
\limsup_n \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|^2 \rightarrow 0.
\]

Since $\{u_n\}$ is bounded in $H_0$, by the Lions lemma it follows that

\[
u_n \rightarrow 0 \quad \text{in } L^r(\mathbb{R}^3), \ 2 < r \leq 6.
\]

From the subcritical growth of $f$, we have

\[
\int_{\mathbb{R}^3} F(u_n^2) \ dx = o_n(1) = \int_{\mathbb{R}^3} f(u_n^2) u_n^2 \ dx.
\]

Moreover, from $I_0'(u_n)[u_n] \rightarrow 0$, it follows that

\[
\int_{\mathbb{R}^3} (a|\nabla u_n|^2 + V_0 u_n^2) \ dx + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \ dx \right)^2 = \int_{\mathbb{R}^3} f(u_n^2) u_n^2 \ dx + o_n(1) = o_n(1).
\]

Thus (i) holds. \hfill $\Box$

Remark 4.4. From Lemma 4.3 we see that if $u$ is the weak limit of a $(PS)_{c_{V_0}}$ sequence $\{u_n\}$ of the functional $I_0$, then we have $u \neq 0$. Otherwise, we have that $u_n \rightarrow 0$ and if $u_n \not\rightarrow 0$, from Lemma 4.3 it follows that there are a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

\[
\liminf_n \int_{B_R(y_n)} |u_n|^2 \geq \beta > 0.
\]

Then set $v_n(x) = u_n(x + z_n)$. It is easy to see that $\{v_n\}$ is also a $(PS)_{c_{V_0}}$ sequence for the functional $I_0$, it is bounded, and there exists $v \in H_0$ such that $v_n \rightarrow v$ in $H_0$ with $v \neq 0$.
Lemma 4.5. Assume that $V_0 > 0$ and $f$ satisfies (f1)-(R4). Then problem (4.1) has a positive ground state solution.

Proof. First of all, it is easy to show that $c_{V_0} > 0$. Moreover, if $u_0 \in \mathcal{N}_0$ satisfies $I_0(u_0) = c_{V_0}$, then $m^{-1}(u_0) \in S_0$ is a minimizer of $\Psi_0$, so that $u_0$ is a critical point of $I_0$ by Lemma 4.2. Now, we show that there exists a minimizer $u \in \mathcal{N}_0$ of $I_0|_{\mathcal{N}_0}$. Since $\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0 = c_{V_0}$ and $S_0$ is a $C^1$ manifold, by Ekeland’s variational principle, there exists a sequence $\omega_n \in S_0$ with $\Psi_0(\omega_n) \rightarrow c_{V_0}$ and $\Psi'_0(\omega_n) \rightarrow 0$ as $n \rightarrow \infty$. Put $u_n = m(\omega_n) \in \mathcal{N}_0$ for $n \in N$. Then $I_0(u_n) \rightarrow c_{V_0}$ and $I'_0(u_n) \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 4.2(b3). Similar to the proof of Lemma 3.4, it is easy to know that $\{u_n\}$ is bounded in $H_0$. Thus, we have $u_n \rightharpoonup u$ in $H_0$, $u_n \rightarrow u$ in $L^1_{lo} (\mathbb{R}^3)$, $1 \leq r < 6$ and $u_n \rightarrow u$ a.e. in $\mathbb{R}^3$, and thus $I'_0(u) = 0$. From Remark 4.4, we know that $u \neq 0$. Moreover,

$$c_{V_0} \leq I_0(u) = I_0(u) - \frac{1}{\theta} I'_0(u)[u]$$

$$= \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2_{L^1} + \left( \frac{1}{4} - \frac{1}{\theta} \right) b \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right)^2 + \frac{1}{\theta} \int_{\mathbb{R}^3} \left( \frac{1}{2} f(u^2)u^2 - \frac{1}{2} F(u^2) \right) \, dx$$

$$\leq \liminf_n \left\{ \left( \frac{1}{2} - \frac{1}{\theta} \right) \|u_n\|^2_{L^1} + \left( \frac{1}{4} - \frac{1}{\theta} \right) b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \, dx \right)^2 + \frac{1}{\theta} \int_{\mathbb{R}^3} \left( \frac{1}{2} f(u_n^2)u_n^2 - \frac{1}{2} F(u_n^2) \right) \, dx \right\}$$

$$= \liminf_n \left\{ I_0(u_n) - \frac{1}{\theta} I'_0(u_n)[u_n] \right\}$$

$$= c_{V_0}.$$

Thus, $u$ is a ground state solution. From the assumption on $f$, we have $u \geq 0$, and thus $u(x) > 0$ for all $x \in \mathbb{R}^3$. The proof is complete. \qed

Arguing as in [5, Proposition 4], there exists a positive radial ground state solution of problem (4.1), which implies that this solution decays exponentially at infinity with its gradient; moreover, this ground state solution is of class $\mathcal{C}^2(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3, \mathbb{R})$.

Lemma 4.6. Let $(u_n) \subset \mathcal{N}_0$ be such that $I_0(u_n) \rightarrow c_{V_0}$. Then $(u_n)$ has a convergent subsequence in $H_0$.

Proof. Since $(u_n) \subset \mathcal{N}_0$, from Lemma 4.1(a3), Lemma 4.2(b4) and the definition of $c_{V_0}$, we have

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0 \quad \text{for all } n \in N,$$

and

$$\Psi_0(v_n) = I_0(u_n) \rightarrow c_{V_0} = \inf_{u \in S_0} \Psi_0(u).$$

Since $S_0$ is a complete $C^1$ manifold, by the Ekeland’s variational principle, there exists a sequence $\{\tilde{v}_n\} \subset S_0$ such that $\{\tilde{v}_n\}$ is a (PS)$_{c_{V_0}}$ sequence for $\Psi_0$ on $S_0$ and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Similar to Lemma 4.5, we may obtain the conclusion of this lemma. \qed

4.2 The Technical Results

In this subsection, we prove a multiplicity result for the modified problem (3.1) using the Ljusternik–Schnirelmann category theory. In order to get it, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, let $\omega \in H^2(\mathbb{R}^3, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and let $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq \delta$, and $\eta(t) = 0$ if $t \geq \delta$. 
For any $y \in M$, let us introduce the function
\[
\Psi_{\epsilon,y}(x) := \eta(|\epsilon x - y|) \omega\left(\frac{\epsilon x - y}{\epsilon}\right) \exp(\imath \tau_y\left(\frac{\epsilon x - y}{\epsilon}\right)),
\]
where
\[
\tau_y(x) := \sum_{i=1}^{3} A_i(y)x_i.
\]
Let $t_\epsilon > 0$ be the unique positive number such that
\[
\max_{t \leq 0} J_\epsilon(t\Psi_{\epsilon,y}) = J_\epsilon(t_\epsilon \Psi_{\epsilon,y}).
\]
Note that $t_\epsilon \Psi_{\epsilon,y} \in N_\epsilon$.

Let us define $\Phi_\epsilon : M \to N_\epsilon$ by
\[
\Phi_\epsilon(y) := t_\epsilon \Psi_{\epsilon,y}.
\]
By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$. Moreover, the energy of the above functions has the following behavior as $\epsilon \to 0^+$.

**Lemma 4.7.** The limit
\[
\lim_{\epsilon \to 0^+} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}
\]
holds uniformly in $y \in M$.

**Proof.** Assume by contradiction that the statement is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0^+$ satisfying
\[
|J_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - c_{V_0}| \geq \delta_0.
\]
For simplicity, we write $\Phi_n$, $\Psi_n$ and $t_n$ for $\Phi_{\epsilon_n}(y_n)$, $\Psi_{\epsilon_n,y_n}$ and $t_{\epsilon_n}$, respectively.

By the Lebesgue dominated convergence theorem, we have that
\[
\|\Psi_n\|_{\epsilon_n}^2 \to \int (|\nabla \omega|^2 + V_0 \omega^2) \, dx \quad \text{as} \quad n \to +\infty, \quad (4.2)
\]
\[
[\Psi_n]_{A_{\epsilon_n}}^\delta \to \int (\omega^4) \quad \text{as} \quad n \to +\infty. \quad (4.3)
\]
Since $J'_{\epsilon_n}(t_n \Psi_n)(t_n \Psi_n) = 0$, by the change of variables $z = (\epsilon_n x - y_n)/\epsilon_n$, observe that, if $z \in B_{B/\epsilon_n}(0)$, then $\epsilon_n z + y_n \in B_B(y_n) \subset M_B \subset \Lambda$. We have
\[
\|\Psi_n\|_{\epsilon_n}^2 + t_n^2 b[\Psi_n]_{A_{\epsilon_n}}^\delta = \int_{\mathbb{R}^3} g(\epsilon_n z + y_n, t_n^2 \eta^2(|\epsilon_n z|) \omega^2(z)) \eta^2(|\epsilon_n z|) \omega^2(z) \, dz
\]
\[
\geq \int_{B_{B/|2\epsilon_n|}(0)} f(t_n^2 \eta^2(|\epsilon_n z|) \omega^2(z)) \omega^2(z) \, dz
\]
\[
\geq \int_{B_{B/2}(0)} f(t_n^2 \omega^2(z)) \omega^2(z) \, dz
\]
\[
\geq f(t_n^2 y^2) \int_{B_{B/2}(0)} \omega^4(z) \, dz
\]
for all $n$ large enough and where $y = \min[\omega(z) : |z| \leq \frac{\delta}{2}]$. Moreover, we have
\[
t_n^2 \|\Psi_n\|_{\epsilon_n}^2 + b[\Psi_n]_{A_{\epsilon_n}}^\delta \geq \frac{f(t_n^2 y^2)^2}{t_n^2 y^2} \int_{B_{B/2}(0)} \omega^4(z) \, dz.
\]
If $t_n \to +\infty$, by (f4) we derive a contradiction.
Therefore, up to a subsequence, we may assume that \( t_n \to t_0 \geq 0 \). If \( t_n \to 0 \), using the fact that \( f \) is increasing and using the Lebesgue dominated convergence theorem, we obtain that

\[
\|\Psi_n\|_{e_n}^2 + t_n^2 b(\Psi_n) \frac{\lambda_n}{4} = \int_{\mathbb{R}^3} f(t_n^2 \eta^2(|\epsilon_n z|) \omega^2(z)) \eta^2(|\epsilon_n z|) \omega^2(z) \, dz \to 0 \quad \text{as} \quad n \to +\infty,
\]

which contradicts (4.2). Thus, from (4.2) and (4.3), we have \( t_0 > 0 \) and

\[
\int_{\mathbb{R}^3} (|\nabla \omega|^2 + V_0 \omega^2) \, dx + t_0^2 b[\omega] \frac{\lambda}{4} = \int_{\mathbb{R}^3} f(t_0 \omega^2) \omega^2 \, dx,
\]

so that \( t_0 \omega \in N_{V_0} \). Since \( \omega \in N_{V_0} \), we obtain that \( t_0 = 1 \) and so, using the Lebesgue dominated convergence theorem, we get

\[
\lim_n \int_{\mathbb{R}^3} F\left(\Psi_n\right)^2 \, dx = \int_{\mathbb{R}^3} F(\omega^2) \, dx.
\]

Hence

\[
\lim_n J_{e_n}(\Phi_{e_n}(y_n)) = I_0(\omega) = c_{V_0},
\]

which is a contradiction and the proof is complete.

Now we define the barycenter map.

Let \( \rho > 0 \) be such that \( M_\delta \subset B_\rho \) and consider \( \Upsilon : \mathbb{R}^3 \to \mathbb{R}^3 \) defined by setting

\[
\Upsilon(x) := \begin{cases}
  x & \text{if } |x| < \rho, \\
  \rho x & \text{if } |x| \geq \rho.
\end{cases}
\]

The barycenter map \( \beta_e : N_e \to \mathbb{R}^3 \) is defined by

\[
\beta_e(u) := \frac{1}{\|u\|_4^4} \int_{\mathbb{R}^3} \Upsilon(\epsilon x)|u(x)|^4 \, dx.
\]

We have the following lemma.

**Lemma 4.8.** The limit

\[
\lim_{\epsilon \to 0^+} \beta_e(\Phi_e(y)) = y
\]

holds uniformly in \( y \in M \).

**Proof.** Assume by contradiction that there exist \( \kappa > 0, (y_n) \subset M \) and \( \epsilon_n \to 0 \) such that

\[
|\beta_{e_n}(\Phi_{e_n}(y_n)) - y_n| \geq \kappa. \tag{4.4}
\]

Using the change of variable \( z = (\epsilon_n x - y_n)/\epsilon_n \), we can see that

\[
\beta_{e_n}(\Phi_{e_n}(y_n)) = y_n + \frac{1}{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\Upsilon(\epsilon_n z + y_n)\eta^4(|\epsilon_n z|)\omega^4(z)}{\eta^4(|\epsilon_n z|)\omega^4(z)} \, dz.
\]

Taking into account \( (y_n) \subset M \subset M_\delta \subset B_\rho \) and the Lebesgue dominated convergence theorem, we can obtain that

\[
|\beta_{e_n}(\Phi_{e_n}(y_n)) - y_n| = o_n(1),
\]

which contradicts (4.4). \( \square \)

Now, we prove the following useful compactness result.
Proposition 4.9. Let $e_n \to 0^+$ and $(u_n) \in \mathcal{N}_{e_n}$ be such that $J_{e_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that the sequence $(|v_n|) \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^3, \mathbb{R})$. Moreover, up to a subsequence, $y_n := e_n\tilde{y}_n \to y \in M$ as $n \to +\infty$.

Proof. Since $J'_{e_n}(u_n)[u_n] = 0$ and $J_{e_n}(u_n) \to c_{V_0}$, arguing as in Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{e_n} \leq C$ for all $n \in \mathbb{N}$.

Arguing as in the proof of Lemma 3.2 and recalling that $c_{V_0} > 0$, we have that there exist a sequence $\{y_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_{n} \int_{B_R(y_n)} |u_n|^2 \, dx \geq \beta. \quad (4.5)$$

Now, let us consider the sequence $|v_n| \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$. By the diamagnetic inequality (2.1), we get that $|v_n|$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R})$. Using (4.5), we may assume that $|v_n| \to \nu$ in $H^1(\mathbb{R}^3, \mathbb{R})$ for some $\nu \neq 0$.

Let $t_n > 0$ be such that $\nu_n := t_nv_n \in \mathcal{N}_{V_0}$, and set $y_n := e_n\tilde{y}_n$.

By the diamagnetic inequality (2.1), we have

$$c_{V_0} \leq I_0(\nu_n) \leq \max_{t \in \mathbb{R}} J_{e_n}(tu_n) = J_{e_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_0(\nu_n) \to c_{V_0}$ as $n \to +\infty$.

Since the sequences $\{|v_n|\}$ and $\{|v_n|\}$ are bounded in $H^1(\mathbb{R}^3, \mathbb{R})$ and $|v_n| \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, we have that $(t_n)$ is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.

We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $|v_n|$ is bounded, we have $\nu_n \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, that is, $I_0(\nu_n) \to 0$, which contradicts $c_{V_0} > 0$.

Thus, up to a subsequence, we may assume that $\nu_n \to \nu := t_0\nu \neq 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, and, by Lemma 4.6, we can deduce that $\nu_n \to \nu$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which gives $|v_n| \to \nu$ in $H^1(\mathbb{R}^3, \mathbb{R})$.

Now we show the final part, namely that $|y_n|$ has a subsequence such that $|y_n| \to y \in M$. Assume by contradiction that $|y_n|$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then, for $n$ large enough, we have $|y_n| > 2R$, and, for any $x \in B_{2R}(0)$,

$$|e_n x + y_n| \geq |y_n| - |e_n x| > R.$$ 

Since $u_n \in \mathcal{N}_{e_n}$, using (V1) and the diamagnetic inequality (2.1), we get that

$$\int_{\mathbb{R}^3} \left( (a|\nabla|v_n|^2 + V_0|v_n|^2) \right) \, dx \leq \int_{\mathbb{R}^3} g(e_n x + y_n, |v_n|^2) |v_n|^2 \, dx$$

$$\leq \int_{B_{2R}(0)} \hat{f}(|v_n|^2) |v_n|^2 \, dx + \int_{B_{2R}(0)} f(|v_n|^2) |v_n|^2 \, dx. \quad (4.6)$$

Since $|v_n| \to \nu$ in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\hat{f}(t) \leq V_0/K$, we can see that (4.6) yields

$$\min \left\{ 1, V_0 \left( 1 - \frac{1}{R} \right) \right\} \int_{\mathbb{R}^3} (a|\nabla|v_n|^2 + V_0|v_n|^2) \, dx = o_n(1),$$

that is, $|v_n| \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which contradicts to $\nu \neq 0$.

Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^3$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every $n$ large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}$. Then, if $x \in B_{2r}(0)$, we have that $|e_n x + y_n - y_0| < 2r$ so that $e_n x + y_n \notin \overline{\Lambda}$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$. 

To prove that \( V(y_0) = V_0 \), we suppose by contradiction that \( V(y_0) > V_0 \). Using Fatou’s lemma, the change of variable \( z = x + \tilde{y}_n \) and \( \max_{t \geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) \), we obtain

\[
c_{V_0} = I_0(\psi) < \frac{1}{2} \int_\mathbb{R}^3 \left( a|\nabla\psi|^2 + V(y_0)|\psi|^2 \right) dx + \frac{b}{4} \left( \int |\nabla\psi|^2 dx \right) - \frac{1}{2} \int F(|\psi|^2) dx
\]

\[
\leq \liminf_n \left( \frac{1}{2} \int_\mathbb{R}^3 \left( a|\nabla\psi_n|^2 + V(\epsilon_n x + y_n)|\psi_n|^2 \right) dx + \frac{b}{4} \left( \int |\nabla\psi_n|^2 dx \right) - \frac{1}{2} \int F(|\psi_n|^2) dx \right)
\]

\[
= \liminf_n \left( \frac{1}{2} \int_\mathbb{R}^3 \left( a|\nabla\psi_n|^2 + V(\epsilon_n x + y_n)|\psi_n|^2 \right) dx + \frac{b}{4} \left( \int |\nabla\psi_n|^2 dx \right) - \frac{1}{2} \int F(|\psi_n|^2) dx \right)
\]

\[
\leq \liminf_n J_{\epsilon_n}(u_n) = c_{V_0},
\]

which is impossible and the proof is complete.

Let now \( \tilde{N}_\epsilon := \{ u \in N_\epsilon : J_\epsilon(u) \leq c_{V_0} + h(\epsilon) \} \), where \( h : \mathbb{R}^+ \to \mathbb{R}^+ \), \( h(\epsilon) \to 0 \) as \( \epsilon \to 0^+ \).

For fixed \( y \in M \), since, by Lemma 4.7, \( |J_\epsilon(\Phi_\epsilon(y))| - c_{V_0} \to 0 \) as \( \epsilon \to 0^+ \), we get that \( \tilde{N}_\epsilon \neq \emptyset \) for any \( \epsilon > 0 \) small enough.

We have the following relation between \( \tilde{N}_\epsilon \) and the barycenter map.

**Lemma 4.10.** We have

\[
\lim_{\epsilon \to 0^+} \sup_{u \in \tilde{N}_\epsilon} \text{dist}(\beta_\epsilon(u), M_\delta) = 0.
\]

**Proof.** Let \( \epsilon_n \to 0^+ \) as \( n \to +\infty \). For any \( n \in \mathbb{N} \), there exists \( u_n \in \tilde{N}_{\epsilon_n} \) such that

\[
\sup_{u \in \tilde{N}_{\epsilon_n}} \inf_{y \in M_\delta} |\beta_{\epsilon_n}(u) - y| = \inf_{y \in M_\delta} |\beta_{\epsilon_n}(u_n) - y| + o_n(1).
\]

Therefore, it is enough to prove that there exists \( (y_n) \subset M_\delta \) such that

\[
\lim_n |\beta_{\epsilon_n}(u_n) - y_n| = 0.
\]

By the diamagnetic inequality (2.1), we can see that \( I_0(tu_n) \leq J_{\epsilon_n}(tu_n) \) for any \( t \geq 0 \). Therefore, recalling that \( \{u_n\} \subset \hat{N}_{\epsilon_n} \subset N_\epsilon \), we can deduce that

\[
c_{V_0} \leq \max_{t \geq 0} I_0(tu_n) \leq \max_{t \geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) \leq c_{V_0} + h(\epsilon_n),
\]

which implies that \( J_{\epsilon_n}(u_n) \to c_{V_0} \) as \( n \to +\infty \). Then Proposition 4.9 implies that there exists \( (\tilde{y}_n) \subset \mathbb{R}^3 \) such that \( y_n = \epsilon_n \tilde{y}_n \in M_\delta \) for \( n \) large enough.

Thus, making the change of variable \( z = x - \tilde{y}_n \), we get

\[
\beta_{\epsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3} (\Phi(\epsilon_n z + y_n) - \Phi(y_n))u_n(z + \tilde{y}_n)^{1/3} dz}{\int_{\mathbb{R}^3} |u_n(z + \tilde{y}_n)|^{1/3} dz}.
\]

Since, up to a subsequence, \( |u_n|(\cdot + \tilde{y}_n) \) converges strongly in \( H^1(\mathbb{R}^3, \mathbb{R}) \) and \( \epsilon_n z + y_n \to y \in M \) for any \( z \in \mathbb{R}^3 \), we conclude the proof. \( \square \)
4.3 Multiplicity of Solutions for Problem (3.1)

Finally, we present a relation between the topology of \( M \) and the number of solutions of the modified problem (3.1).

**Theorem 4.11.** For any \( \delta > 0 \) such that \( M_\delta \subset \Lambda \), there exists \( \tilde{\delta} > 0 \) such that, for any \( \varepsilon \in (0, \tilde{\delta}) \), problem (3.1) has at least \( \text{cat}_{M_\delta}(M) \) nontrivial solutions.

**Proof.** For any \( \varepsilon > 0 \), we define the function \( \pi_\varepsilon : M \to S^e \) by

\[
\pi_\varepsilon(y) = m^{-1}_\varepsilon(\Phi_\varepsilon(y)) \quad \text{for all } y \in M.
\]

By Lemma 4.7 and Lemma 3.3 (B4), we obtain

\[
\lim_{\varepsilon \to 0} \Psi(\pi_\varepsilon(y)) = \lim_{\varepsilon \to 0} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0} \quad \text{uniformly in } y \in M.
\]

Hence, there is a number \( \hat{\delta} > 0 \) such that the set \( S^e_{\hat{\delta}} := \{ u \in S^e : \Psi(\varepsilon(u) \leq c_{V_0} \varepsilon + h(\varepsilon) \} \) is nonempty for all \( \varepsilon \in (0, \hat{\delta}) \) since \( \pi_\varepsilon(M) \subset S^e_{\hat{\delta}} \). Here \( h \) is given in the definition of \( N_\varepsilon \).

Given \( \delta > 0 \), by Lemma 4.7, Lemma 3.2 (A3), Lemma 4.8, and Lemma 4.10, we can find \( \tilde{\delta} > 0 \) such that for any \( \varepsilon \in (0, \tilde{\delta}) \) the diagram

\[
M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m^{-1}_\varepsilon} \pi_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta
\]

is well defined and continuous. From Lemma 4.8, we can choose a function \( \Theta(\varepsilon, z) \) with \( |\Theta(\varepsilon, z)| < \frac{\hat{\delta}}{2} \) uniformly in \( z \in M \) for all \( \varepsilon \in (0, \hat{\delta}) \) such that \( \beta_\varepsilon(\Phi_\varepsilon(z)) = z + \Theta(\varepsilon, z) \) for all \( z \in M \). Define

\[
H(t, z) = z + (1 - t) \Theta(\varepsilon, z).
\]

Then \( H : [0, 1] \times M \to M_\delta \) is continuous. Clearly, \( H(0, z) = \beta_\varepsilon(\Phi_\varepsilon(z)) \) and \( H(1, z) = z \) for all \( z \in M \). That is, \( H(t, z) \) is a homotopy between \( \beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \pi_\varepsilon \) and the embedding \( \iota : M \to M_\delta \). Thus, this fact implies that

\[
\text{cat}_{\pi_\varepsilon(M)}(\pi_\varepsilon(M)) \geq \text{cat}_{M_\delta}(M).
\]

By Corollary 3.6 and the abstract category theorem [21], \( \Psi_\varepsilon \) has at least \( \text{cat}_{\pi_\varepsilon(M)}(\pi_\varepsilon(M)) \) critical points on \( S^e_\delta \). Therefore, from Lemma 3.3 (B4) and (4.7), we have that \( J_\varepsilon \) has at least \( \text{cat}_{M_\delta}(M) \) critical points in \( N_\varepsilon \), which implies that problem (3.1) has at least \( \text{cat}_{M_\delta}(M) \) solutions.

5 Proof of Theorem 1.1

In this section, we prove our main result. The idea is to show that the solutions \( u_\varepsilon \) obtained in Theorem 4.11 satisfy

\[
|u_\varepsilon(x)|^2 \leq a_0 \quad \text{for } x \in \Lambda_\varepsilon^C
\]

for \( \varepsilon > 0 \) small. Arguing as in [26], the following uniform result holds.

**Lemma 5.1.** Let \( \varepsilon_n \to 0^+ \) and let \( u_n \in N_\varepsilon \) be a solution of problem (3.1) for \( \varepsilon = \varepsilon_n \). Then \( u_n \rightharpoonup u \) in \( c_{V_0} \). Moreover, there exists \( \tilde{y}_n \subset \mathbb{R}^3 \) such that, if \( v_n(x) := u_n(x + \tilde{y}_n) \), we have that \( |v_n| \) is bounded in \( L^{\infty}(\mathbb{R}^3, \mathbb{R}) \) and

\[
\lim_{n \to \infty} |v_n(x)| = 0 \quad \text{uniformly in } n \in \mathbb{N}.
\]

Now, we are ready to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \delta > 0 \) be such that \( M_\delta \subset \Lambda \). We want to show that there exists \( \tilde{\delta} > 0 \) such that for any \( \varepsilon \in (0, \tilde{\delta}) \) and any solution \( u_\varepsilon \in N_\varepsilon \) of problem (3.1), it holds

\[
\|u_\varepsilon\|^2_{L^{\infty}(\Lambda_\varepsilon)} \leq a_0.
\]

(5.1)
We argue by contradiction and assume that there is a sequence \(c_n \to 0\) such that for every \(n\) there exists \(u_n \in \tilde{N}_{\varepsilon_n}\) which satisfies \(I_{\varepsilon_n}(u_n) = 0\) and
\[
\|u_n\|_{L^\infty(B_{R_n}(y_n))} > a_0.
\]
As in Lemma 5.1, we have that \(I_{\varepsilon_n}(u_n) \to c_{V_0}\), and therefore we can use Proposition 4.9 to obtain a sequence \((\tilde{y}_n) \subset \mathbb{R}^3\) such that \(y_n := u_n \tilde{y}_n \to y_0\) for some \(y_0 \in M\). Then we can find \(r > 0\) such that \(B_r(y_n) \subset \Lambda\), and so \(B_{r/\varepsilon_n}(y_n) \subset \Lambda_{\varepsilon_n}\) for all \(n\) large enough.

By using Lemma 5.1, there exists \(R > 0\) such that \(|\nabla u_n|^2 \leq a_0\) in \(B_{r/\varepsilon_n}(0)\) and \(n\) large enough, where \(u_n = u_n(\cdot + \tilde{y}_n)\). Hence \(|u_n|^2 \leq a_0\) in \(B_{r/\varepsilon_n}(\tilde{y}_n)\) and \(n\) large enough. Moreover, if \(n\) is so large that \(r/\varepsilon_n > R\), then
\[
\Lambda_{\varepsilon_n} \subset B_{r/\varepsilon_n}(\tilde{y}_n) \subset B_{r/\varepsilon_n}(y_n),
\]
which gives \(|u_n|^2 \leq a_0\) for any \(x \in \Lambda_{\varepsilon_n}\). This contradicts (5.2) and proves the claim.

Let now \(\varepsilon_0 := \min\{\varepsilon_0, \varepsilon_\delta\}\), where \(\varepsilon_\delta \geq 0\) is given by Theorem 4.11. Then we have \(\kappa_\delta(M)\) nontrivial solutions to problem (3.1). If \(u_{\varepsilon} \in \tilde{N}_{\varepsilon}\) is one of these solutions, then, by (5.1) and the definition of \(g\), we conclude that \(u_{\varepsilon}\) is also a solution to problem (2.2).

Finally, we study the behavior of the maximum points of \(|\tilde{u}_{\varepsilon}|\), where \(\tilde{u}_{\varepsilon}(x) := u_{\varepsilon}(\frac{x}{\varepsilon})\) is a solution to problem (1.1) as \(\varepsilon \to 0^+\).

Take \(c_n \to 0^+\) and the sequence \((u_n)\) where each \(u_n\) is a solution of (3.1) for \(\varepsilon = c_n\). From the definition of \(g\), there exists \(y \in (0, a)\) such that
\[
g(\varepsilon x, t^2)|t^2| \leq \frac{V_0}{K} t^2 \quad \text{for all} \quad x \in \mathbb{R}^N, \ |t| \leq y.
\]
Arguing as above, we can take \(R > 0\) such that, for \(n\) large enough,
\[
\|u_n\|_{L^\infty(B_{R}(y_n))} < y.
\]
Up to a subsequence, we may also assume that, for \(n\) large enough,
\[
\|u_n\|_{L^\infty(B_{R}(y_n))} \geq y.
\]
Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have \(\|u_n\|_{\infty} < y\). Thus, since \(I'_{\varepsilon_n}(u_{\varepsilon_n}) = 0\), using (g5) and the diamagnetic inequality (2.1), we obtain
\[
\int_{\mathbb{R}^3} \left( a|\nabla u_n|^2 + V_0|u_n|^2 \right) dx + b \int_{\mathbb{R}^3} \left( |\nabla|u_n|^2 \right) dx \leq \int_{\mathbb{R}^3} g(\varepsilon x, |u_n|^2)|u_n|^2 dx \leq \frac{V_0}{K} \int_{\mathbb{R}^3} |u_n|^2 dx.
\]
Since \(K > 2\), we obtain \(\|u_n\| = 0\), which is a contradiction.

Taking into account (5.3) and (5.4), we can infer that the global maximum points \(p_n\) of \(|\tilde{u}_{\varepsilon_n}|\) belong to \(B_R(\tilde{y}_n)\), that is, \(p_n = q_n + \tilde{y}_n\) for some \(q_n \in B_R\). Recalling that the associated solution of problem (1.1) is \(\tilde{u}_n(x) = \tilde{u}_n(x/\varepsilon_n)\), we can see that a maximum point \(\eta_{\varepsilon_n}\) of \(|u_n|\) is \(\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n\). Since \(q_n \in B_R\), \(\varepsilon_n \tilde{y}_n \to y_0\) and \(V(y_0) = V_0\), the continuity of \(V\) allows to conclude that
\[
\lim_{\varepsilon_n} V(\eta_{\varepsilon_n}) = V_0.
\]
The proof is now complete.

\[\square\]

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