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Picard and Krasnoselski sequences: applications to fixed point problems\textsuperscript{1}

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\textbf{Abstract.} We are concerned with the role of Picard and Krasnoselski sequences in the approximation of fixed points in various classes of nonlinear equations. We also give a connection with the cobweb method that describes equilibrium phenomena in mathematical economics.

\textbf{Keywords:} fixed point; Brouwer fixed point theorem; Knaster fixed point theorem; successive approximation; Picard sequence; Krasnoselski sequence; asymptotic regularity; cobweb method.

\textbf{MSC} : 26A15; 26A18; 47H10.

1. Introduction

The development of fixed point theory is closely related with the study of various problems arising in the theory of ordinary differential equations. One of the first contributions to this field is due to the French mathematician Henri Poincaré\textsuperscript{4} (1854–1912) in his famous paper [14] of 1890 on the three-body problem crowned by \textit{King Oscar Prize}. This problem concerns the free

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\textsuperscript{4}One of the most influential mathematician of modern times, with crucial contributions to the development of applied mathematics, mathematical physics, and celestial mechanics. According to H. Brezis and F. Browder [3], "Poincaré emphasized that a wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance – un “air de famille” in Poincaré’s words – and should be treated by common methods".
motion of multiple orbiting bodies and Poincaré reduced the study to the qualitative analysis of the $T$–periodic solutions of a differential system in $\mathbb{R}^n$

$$x' = f(t, x)$$ (1)

to the study of the fixed points of the operator $P_T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$P_T(y) = p(T; 0, y),$$

where $p(t; s, y)$ denotes the solution of equation (1) verifying the initial condition $x(s) = y$. We refer to the survey paper by Mawhin [11] for more details and related results.

As early as 1883, Poincaré stated in [13] a theorem shown much later to be equivalent with a fixed point theorem for continuous functions on a closed ball into itself, published by L.E.J. Brouwer\textsuperscript{1)} in 1912 (see [4]). In its simplest one-dimensional case (see Figure ), the Brouwer fixed point theorem asserts the following property: any continuous function $f : [a, b] \to [a, b]$ has at least a fixed point. The proof combines very simple arguments, which strongly rely on the main continuity assumption combined with the order structure of the set of real numbers. In the general form, the Brouwer fixed point theorem asserts that any continuous function with domain the closed unit ball $B$ in $\mathbb{R}^N$ and range contained in $B$ must have at least one fixed point. This result was first applied in 1943 to some forced Liénard equations by Lefschetz [9] and Levinson [10]. If $N \geq 2$, the proof of the the Brouwer theorem is much more complicated. However, simpler proofs have been found by means of powerful topological tools, such as the topological degree.

In this paper we are concerned with the following natural related questions:

– what happens if the continuity hypothesis in the Brouwer fixed point theorem is replaced with a monotonicity assumption;

– what about the approximation of the fixed point by means of two classical successive approximations:

$$x_{n+1} = f(x_n) \quad (\text{Picard sequence})$$

\textsuperscript{1)}Dutch mathematician (1881–1966).
\[ x_{n+1} = \frac{x_n + f(x_n)}{2} \] (Krasnoselski sequence).

If \( f \) is continuous, these sequences provide fixed points, provided that they are convergent. Indeed, taking the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 1 - x \), let us consider \( x_0 \in [0, 1] \) and \( x_{n+1} = f(x_n) \), for all \( n \geq 0 \). Then the Picard sequence \( (x_n) \) converges if and only if \( x_0 = 1/2 \). However, if we construct the Krasnoselski sequence \( x_{n+1} = \frac{x_n + f(x_n)}{2} \), then \( (x_n) \) converges for any initial value \( x_0 \in [0, 1] \).

We also establish in the present paper related fixed point properties. A particular interest is given to the cobweb method arising in mathematical economics, in strong relationship with successive approximations. We refer to the recent problem books [15], [16] for further results and relevant applications.

2. Knaster fixed point theorem

In this section we argue that the fixed point property stated in the Brouwer theorem remains true if the continuity assumption is replaced with the hypothesis that the function \( f : [a, b] \to [a, b] \) is non-decreasing. The same property does not hold provided that if \( f \) is decreasing. In the non-decreasing case, the proof strongly relies on the order structure of the real axis.

**Theorem 1.** Let \( f : [a, b] \to [a, b] \) be a non-decreasing function.

(i) Then \( f \) has at least one fixed point.

(ii) There are decreasing function \( f : [a, b] \to [a, b] \) with no fixed points.

**Proof.** (i) Set
\[ A = \{ a \leq x \leq b; \ f(x) \geq x \} \]
and \( x_0 = \sup A \). The following situations may occur.

![Graph showing Knaster's fixed point theorem](image)

**Fig. 2.** Knaster’s fixed point theorem

**Case 1:** \( x_0 \in A \). By the definition of \( x_0 \) it follows that \( f(x_0) \geq x_0 \). If \( f(x_0) = x_0 \), then the proof is concluded. If not, we argue by contradiction and assume that \( f(x_0) > x_0 \). By the definition of \( x_0 \) we obtain \( f(x) < x; \ \forall x > x_0 \).
On the other hand, for any $x_0 < x < f(x_0)$ we have $x > f(x) > f(x_0)$, contradiction since $x \in (x_0, f(x_0))$, that is, $x < f(x)$. It follows that the assumption $f(x_0) > x_0$ is false, so $f$ has a fixed point.

Case 2: $x_0 \not\in A$. We prove that, in fact, it is impossible to have $x_0 \not\in A$, so $x_0 \in A$, which reduces the problem to Case 1. If $x_0 \not\in A$ then there exists a sequence $x_n \to x_0$, $x_n < x_0$, such that $x_n \in A$. Since $f$ is increasing, it follows that $\lim_{n \to \infty} f(x_n) = x_0$. On the other hand, from $f(x_0) < x_0$ we deduce that there exists $x_n < x_0$ such that $f(x_n) > f(x_0)$, contradiction with the fact that $f$ is increasing.

(ii) Consider the function

$$f(x) = \begin{cases} 
1 - x & \text{if } 0 \leq x < \frac{1}{2} \\
\frac{1}{2} - \frac{x}{2} & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

Then $f : [0, 1] \to [0, 1]$ is decreasing but does not have any fixed point. \hfill $\square$

A related counterexample is depicted in Figure 3.

![Fig. 3. Knaster’s fixed point theorem fails for decreasing functions](image)

3. Further fixed point properties

We start with some simple facts regarding the hypotheses of the Brouwer fixed point theorem on the real axis.

(i) While a fixed point in $[a, b]$ exists for a continuous function $f : [a, b] \to [a, b]$, it need not be unique. Indeed, any point $x \in [a, b]$ is a fixed point of the function $f : [a, b] \to [a, b]$ defined by $f(x) = x$.

(ii) The condition that $f$ is defined on a closed subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = (1 + x)/2$, then $f$ maps $[0, 1]$ into itself, and $f$ is continuous. However, $f$ has no fixed point in $[0, 1]$.

(iii) The condition that $f$ be defined on a bounded subset of $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $f : [1, \infty) \to \mathbb{R}$ is
defined by $f(x) = x + x^{-1}$, then $f$ maps $[1, \infty)$ into itself, $f$ is continuous, but $f$ has no fixed point in $[1, \infty)$.

(iv) The condition that $f$ be defined on an interval in $\mathbb{R}$ is essential for the existence of a fixed point. For example, if $D = [-2, -1] \cup [1, 2]$ and $f : D \to \mathbb{R}$ is defined by $f(x) = -x$, then $f$ maps $D$ into itself, $f$ is continuous, but $f$ has no fixed point in $D$.

We prove in what follows some elementary fixed point properties of real-valued functions.

**Proposition 1.** Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$, $f(1) = 1$. Denote $f^n := f \circ f \circ \cdots \circ f$ ($n$ times) and assume that there exists a positive integer $m$ such that $f^n(x) = x$ for all $x \in [0, 1]$. Then $f(x) = x$ for any $x \in [0, 1]$.

**Proof.** Our hypothesis implies that $f$ is one-to-one, so increasing (since $f$ is continuous). Assume, by contradiction, that there exists $x \in (0, 1)$ such that $f(x) > x$. Then, for any $n \in \mathbb{N}$, we have $f^n(x) > f^{n-1}(x) > \cdots > f(x) > x$. For $n = m$ we find a contradiction. A similar argument shows that the case $f(x) < x$ (for some $x$) is not possible. □

**Proposition 2.** Let $a$, $b$ be real numbers, $a < b$ and consider a continuous function $f : [a, b] \to \mathbb{R}$.

(i) If $[a, b] \subset f([a, b])$ then $f$ has a fixed point.

(ii) Assume that there exists a closed interval $I' \subset f([a, b])$. Then $I' = f(J)$, where $J$ is a closed interval contained in $[a, b]$.

(iii) Assume that there exists $n$ closed intervals $I_0, \ldots, I_{n-1}$ contained in $[a, b]$ such that for all $0 \leq k \leq n-2$, $I_{k+1} \subset f(I_k)$ and $I_0 \subset f(I_{n-1})$. Then $f^n$ has a fixed point ($f^n = f \circ \cdots \circ f$).

**Proof.** (i) Denote $f([a, b]) = [m, M]$ and let $x_m, x_M \in [a, b]$ be such that $f(x_m) = m$ and $f(x_M) = M$. Since $f(x_m) - x_m \leq 0$ and $f(x_M) - x_M \geq 0$, it follows by the intermediate value property that $f$ has at least a fixed point.

(ii) Set $I' = [c, d]$ and consider $u, v \in I$ such that $f(u) = c$ and $f(v) = d$. Assume, without loss of generality, that $u < v$.

The set $A = \{x \in [u, v]; f(x) = c\}$ is compact and non-empty, so there exists $\alpha = \max\{x; x \in A\}$ and, moreover, $\alpha \in A$. Similarly, the set $B = \{x \in [u, v]; f(x) = d\}$ has a minimum point $\beta$. Then $f(\alpha) = c$, $f(\beta) = d$ and for all $x \in (\alpha, \beta)$ we have $f(x) \neq c$ and $f(x) \neq d$. So, by the intermediate value property, $[c, d] \subset f((\alpha, \beta))$ and $f((\alpha, \beta))$ is an interval which contains neither $c$ nor $d$. It follows that $I' = f(J)$, where $J = [\alpha, \beta]$.

(iii) Since $I_0 \subset f(I_{n-1})$, it follows by (b) that there exists a closed interval $J_{n-1} \subset I_{n-1}$ such that $I_0 = f(J_{n-1})$. But $J_{n-1} \subset I_{n-1} \subset f(I_{n-2})$. So, by (ii), there exists a closed interval $J_{n-2} \subset I_{n-2}$ such that $J_{n-1} = f(J_{n-2})$. Thus, we obtain $n$ closed intervals $J_0, \ldots, J_{n-1}$ such that $J_k \subset I_k$, for all $0 \leq k \leq n - 1$.

...
and

\[ J_{k+1} = f(J_k), \quad \text{for all } 0 \leq k \leq n - 2 \text{ and } J_0 = f(J_{n-1}). \]

Consequently, \( J_0 \) is included in the domain of the \( n \)th iterate \( f^n \) and \( J_0 \subset I_0 = f^n(J_0) \). By (a) we deduce that \( f^n \) has a unique fixed point in \( J_0 \).

\[ \Box \]

**Proposition 3.** Suppose that \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( |f(x) - f(y)| < |x - y| \) whenever \( x \neq y \). Then there is some \( \xi \) in \([-\infty, +\infty]\) such that, for any real \( x \), \( f^n(x) \to \xi \) as \( n \to \infty \).

**Proof.** We suppose first that \( f \) has a fixed point, say \( \xi \), in \( \mathbb{R} \). Then, from the contracting property of \( f \), \( \xi \) is the only fixed point of \( f \). We may assume that \( \xi = 0 \), and this implies that \( |f(x)| < |x| \) for all nonzero \( x \).

Thus, for any \( x \), the sequence \( |f^n(x)| \) is decreasing, so converges to some nonnegative number \( \mu(x) \). We want to show that \( \mu(x) = 0 \) for every \( x \), so suppose now that \( x \) is such that \( \mu(x) > 0 \). Then \( f \) maps \( \mu(x) \) and \(-\mu(x)\) to points \( y_1 \) and \( y_2 \), say, where \( y_j < |\mu(x)| \) for each \( j \). Thus, as \( f \) is continuous, there are open neighborhoods of \( \pm \mu(x) \) that are mapped by \( f \) into the open interval \( I = (-\mu(x), \mu(x)) \) that contains \( y_1 \) and \( y_2 \). This implies that, for sufficiently large \( n \), \( f^n(x) \) lies in \( I \), which contradicts the fact that \( |f^n(x)| \geq \mu(x) \) for all \( n \). Thus, for all \( x \), \( \mu(x) = 0 \) and \( f^n(x) \to 0 \).

Now suppose that \( f \) has no fixed point in \( \mathbb{R} \). Then the function \( f(x) - x \) is continuous and nonzero in \( \mathbb{R} \). By the intermediate value theorem, \( f(x) > x \) for all \( x \), or \( f(x) < x \) for all \( x \). We may assume that \( f(x) > x \) for all \( x \), as similar argument holds in the other case. Now the sequence \( f^n(x) \) is strictly increasing, hence converges to some \( \xi \) in \( \mathbb{R} \cup \{+\infty\} \). Moreover, \( \xi \notin \mathbb{R} \), else \( \xi \) would be a fixed point of \( f \). Thus \( f^n(x) \to +\infty \) for all \( x \).

We conclude this paper with the following elementary property, which is due to M. W. Botsko [2].

**Proposition 4.** Let \( f : [0, 1] \to [0, 1] \) be a function such that \( |f(x) - f(y)| \leq |x - y| \) for all \( x, y \in [0, 1] \). Then the set of all fixed points of \( f \) is either a single point or an interval.

**Proof.** Let \( F = \{ x \in [0, 1] : f(x) = x \} \). Since \( F \) is continuous, it follows that \( F \) is compact. Let \( a \) be the smallest number in \( F \) and \( b \) the largest number in \( F \). It follows that \( F \subset [a, b] \). Fix arbitrarily \( x_0 \in [a, b] \).

Since \( a \) is a fixed point of \( f \), we have

\[ f(x_0) - a \leq |f(x_0) - a| = |f(x_0) - f(a)| \leq x_0 - a. \]

Therefore, \( f(x_0) \leq x_0 \). Similarly,

\[ b - f(x_0) \leq |b - f(x_0)| = |f(b) - f(x_0)| \leq b - x_0, \]

which shows that \( f(x_0) \geq x_0 \). It follows that \( f(x_0) = x_0 \), hence \( x_0 \) is a fixed point of \( f \). Thus, \( F = [a, b] \).

\[ \Box \]
4. Approximation of fixed points

We have observed that if $f : [a, b] \to [a, b]$ is a continuous function then $f$ must have at least one fixed point, that is, a point $x \in [a, b]$ such that $f(x) = x$. A natural question in applications is to provide an algorithm for finding (or approximating) this point. One method of finding such a fixed point is by successive approximation. This technique is due to the French mathematician Émile Picard (1856–1941) and was introduced in his classical textbook on analysis [12]. More precisely, if $x_1 \in [a, b]$ is chosen arbitrarily, define $x_{n+1} = f(x_n)$ and the resulting sequence $(x_n)_{n \geq 1}$ is called the sequence of successive approximations of $f$ (or a Picard sequence for the function $f$). If the sequence $(x_n)_{n \geq 1}$ converges to some $x$, then a direct argument based on the continuity of $f$ shows that $x$ is a fixed point of $f$. Indeed,

$$f(x) = f \left( \lim_{n \to \infty} x_n \right) = f \left( \lim_{n \to \infty} x_{n-1} \right) = \lim_{n \to \infty} f(x_{n-1}) = \lim_{n \to \infty} x_n = x.$$ 

The usual method of showing that the sequence $(x_n)_{n \geq 1}$ of successive approximations converges is to show that it satisfies the Cauchy convergence criterion: for every $\varepsilon > 0$ there is an integer $N$, such that for all integers $j, k \geq N$, we have $|x_j - x_k| < \varepsilon$. The next exercise asserts that it is enough to set $j = k + 1$ in the Cauchy criterion.

Proposition 5. Let $f : [a, b] \to [a, b]$ be a continuous function. Let $x_1$ be a point in $[a, b]$ and let $(x_n)_{n \geq 1}$ denote the resulting sequence of successive approximations. Then the sequence $(x_n)_{n \geq 1}$ converges to a fixed point of $f$ if and only if $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$.

Proof. Clearly $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$ if $(x_n)_{n \geq 1}$ converges to a fixed point. Suppose $\lim_{n \to \infty} (x_{n+1} - x_n) = 0$ and the sequence $(x_n)_{n \geq 1}$ does not converge. Since $[a, b]$ is compact, there exist two subsequences of $(x_n)_{n \geq 1}$ that converge to $\xi_1$ and $\xi_2$ respectively. We may assume $\xi_1 < \xi_2$. It suffices to show that $f(x) = x$ for all $x \in (\xi_1, \xi_2)$. Suppose this is not the case, hence there is some $x^* \in (\xi_1, \xi_2)$ such that $f(x^*) \neq x^*$. Then a $\delta > 0$ could be found such that $[x^* - \delta, x^* + \delta] \subset (\xi_1, \xi_2)$ and $f(\bar{x}) \neq \bar{x}$ whenever $\bar{x} \in (x^* - \delta, x^* + \delta)$. Assume $\bar{x} - f(\bar{x}) > 0$ (the proof in the other case being analogous) and choose $N$ so that $|f^n(x) - f^n(\bar{x})| < \delta$ for $n > N$. Since $\xi_2$ is a cluster point, there exists a positive integer $n > N$ such that $f^n(x) > x^*$. Let $n_0$ be the smallest such integer. Then, clearly,

$$f^{n_0-1}(x) < x^* < f^{n_0}(x)$$

and since $f^{n_0}(x) - f^{n_0-1}(x) < \delta$ we must have

$$f^{n_0-1}(x) - f^{n_0}(x) > 0$$

so that $f^{n_0}(x) < f^{n_0-1}(x) < x^*$,

a contradiction. \qed
The usual method of showing that the sequence \((x_n)_{n \geq 0}\) of successive approximations converges is to show that it satisfies the Cauchy convergence criterion. The next result establishes that this happens if and only if the difference of two consecutive terms in this iteration converges to zero. The American mathematician Felix Browder has called this condition asymptotic regularity.

The next result is due to H.G. Barone [1] and was established in 1939.

**Theorem 2.** Let \((x_n)_{n \geq 0}\) be a sequence of real numbers such that the sequence \((x_{n+1} - x_n)\) converges to zero. Then the set of cluster points of \((x_n)_{n \geq 0}\) is a closed interval in \(\mathbb{R}\), eventually degenerated.

**Proof.** Set \(\ell_- := \liminf_{n \to \infty} x_n\), \(\ell_+ := \limsup_{n \to \infty} x_n\) and choose \(a \in (\ell_-, \ell_+).\) By the definition of \(\ell_-\), there exists \(x_{n_1} < a\). Let \(n_2\) be the least integer greater than \(n_1\) such that \(x_{n_2} > a\) (the existence of \(n_2\) follows by the definition of \(\ell_+\)). Thus, \(x_{n_2} \leq a < x_{n_3}\). Since \(\ell_- < a\), there exists a positive integer \(n_3 > n_2\) such that \(x_{n_3} < a\). Next, by the definition of \(\ell_+\), there exists an integer \(N_1 > n_3\) such that \(x_{N_2} > a\). If \(n_4\) denotes the least integer with these properties, then \(x_{n_4} < a < x_{n_4}\). In this manner we construct an increasing sequence of positive numbers \((n_{2k})_{k \geq 1}\) such that, for all \(k \geq 1\), \(x_{n_{2k}} < a < x_{n_{2k+1}}\). Using the hypothesis we deduce that the sequences \((x_{n_{2k}-1})_{k \geq 1}\) and \((x_{n_{2k}})_{k \geq 1}\) converge to \(a\), so \(a\) is a cluster point.

The following convergence result was established by B.P. Hillam [7] in 1976.

**Theorem 3.** Let \(f : [a, b] \to [a, b]\) be a continuous function. Consider the sequence \((x_n)_{n \geq 0}\) defined by \(x_0 \in [a, b]\) and, for any positive integer \(n\), \(x_n = f(x_{n-1})\). Then the sequence \((x_n)_{n \geq 0}\) converges if and only if \((x_{n+1} - x_n)\) converges to zero.

**Proof.** Assume that the sequence of successive approximations \((x_n)_{n \geq 0}\) satisfies \(x_{n+1} - x_n \to 0\) as \(n \to \infty\). With the same notations as above, assume that \(\ell_- < \ell_+\). The proof of (i) combined with the continuity of \(f\) imply \(a = f(a)\), for all \(a \in (\ell_-, \ell_+)\). But this contradicts our assumption \(\ell_- < \ell_+\). Indeed, choose \(\ell_- < c < d < \ell_+\) and \(0 < \varepsilon < (d - c)/3\). Since \(x_{n+1} - x_n \to 0\), there exists \(N_\varepsilon\) such that for all \(n \geq N_\varepsilon\), \(-\varepsilon < x_{n+1} - x_n < \varepsilon\). Let \(N_2 > N_1 > N_\varepsilon\) be such that \(x_{N_1} < c < d < x_{N_2}\). Our choice of \(\varepsilon\) implies that there exists an integer \(n \in (N_1, N_2)\) such that \(a := x_n \in (c, d)\). Hence \(x_{n+1} = f(a) = a, x_{n+2} = a, \) and so on. Therefore \(x_{N_2} = a\), contradiction.

The reversed assertion is obvious.

The following result is a particular case of a fixed point theorem due to Krasnoselskii (see [8]). We refer to [6] for the general framework corresponding to functions defined on a closed convex subset of strictly convex Banach spaces.
Theorem 4. Let \( f : [a, b] \to [a, b] \) a function satisfying \( |f(x) - f(y)| \leq |x - y| \), for all \( x, y \in [a, b] \). Define the sequence \( (x_n)_{n \geq 1} \) by \( x_1 \in [a, b] \) and, for all \( n \geq 1 \), \( x_{n+1} = \frac{|x_n + f(x_n)|}{2} \). Then the sequence \( (x_n)_{n \geq 1} \) converges to some fixed point of \( f \).

Proof. We observe that it is enough to show that \( (x_n)_{n \geq 1} \) converges. In this case, by the recurrence relation and the continuity of \( f \), it follows that the limit of \( (x_n)_{n \geq 1} \) is a fixed point of \( f \). We argue by contradiction and denote by \( A \) the set of all limit points of \( (x_n)_{n \geq 1} \), that is,

\[
A := \{ \ell \in [a, b] \mid \text{there exists a subsequence } (x_{n_k})_{k \geq 1} \text{ of } (x_n)_{n \geq 1} \text{ such that } x_{n_k} \to \ell \}.
\]

By our hypothesis and the compactness of \([a, b]\), we deduce that \( A \) contains at least two elements and is a closed set.

We split the proof into several steps.

(i) For any \( \ell \in A \) we have \( f(\ell) \neq \ell \). Indeed, assume that \( \ell \in A \) and fix \( \epsilon > 0 \) and \( n_k \in \mathbb{N} \) such that \( |x_{n_k} - \ell| \leq \epsilon \). Then

\[
|\ell - x_{n_k + 1}| = \frac{\ell + f(\ell)}{2} - \frac{x_{n_k} + f(x_{n_k})}{2} \leq \frac{|\ell - f(x_{n_k})|}{2} + \frac{|f(\ell) - f(x_{n_k})|}{2} \leq |\ell - x_{n_k}| \leq \epsilon
\]

and so on. This shows that \( |x_n - \ell| \leq \epsilon \), for all \( n \geq n_k \). Hence \( (x_n)_{n \geq 1} \) converges to \( \ell \), contradiction.

(ii) There exists \( \ell_0 \in A \) such that \( f(\ell_0) > \ell_0 \). Indeed, arguing by contradiction, set \( \ell_- = \min \ell \in A \) and \( f(\ell_-) \leq \ell_- \). The variant \( f(\ell_-) = \ell_- \) is excluded, by (i). But \( f(\ell_-) < \ell_- \) implies that \( \frac{\ell_- + f(\ell_-)}{2} \in A \) and \( \frac{\ell_- + f(\ell_-)}{2} < \ell_- \), which contradicts the definition of \( \ell_- \).

(iii) There exists \( \epsilon > 0 \) such that \( |f(\ell) - \ell| \geq \epsilon \), for all \( \ell \in A \). For if not, let \( \ell_n \in A \) such that \( |f(\ell_n) - \ell_n| < \frac{\epsilon}{n} \), for all \( n \geq 1 \). This implies that any limit point of \( (\ell_n)_{n \geq 1} \) (which lies in \( A \), too) is a fixed point of \( f \). This contradicts (i).

(iv) Conclusion. By (ii) and (iii), there exists a largest \( \ell_+ \in A \) such that \( f(\ell_+) > \ell_+ \). Let \( \ell' = \frac{\ell_+ + f(\ell_+)}{2} \) and observe that \( f(\ell_+) > \ell' > \ell_+ \) and \( f(\ell') < \ell' \). By (iii), there exists a smallest \( \ell'' \in A \) such that \( \ell'' > \ell_+ \) and \( f(\ell'') < \ell'' \). It follows that \( \ell_+ < \ell'' < f(\ell_+) \). Next note that \( f(\ell'') < \ell_- \); for, if not, \( \ell''' = \frac{\ell_+ + f(\ell'')}{2} \) satisfies \( \ell_+ < \ell''' < \ell_- \) and, by definitions of \( \ell_+ \) and \( \ell'' \), it follows that \( f(\ell''' = \ell''' \), contrary to (i). Thus \( f(\ell'') < \ell_- < \ell'' < f(\ell_+) \). It then follows that \( |f(\ell'') - f(\ell_+)| > |\ell'' - \ell_-| \). This contradicts the hypothesis and concludes the proof.

Remark. The iteration scheme described in the above Krasnoselski’s property does not apply to arbitrary continuous mappings of a closed interval
into itself. Indeed, consider the function $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 
  \frac{3}{4}, & \text{if } 0 \leq x \leq \frac{1}{4} \\
  -3x + \frac{3}{2}, & \text{if } \frac{1}{4} < x < \frac{1}{2} \\
  0, & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

Then the sequence defined in the above statement is defined by $x_{2n} = \frac{1}{2}$ and $x_{2n+1} = \frac{1}{4}$, for any $n \geq 1$. So, $(x_n)_{n\geq1}$ is a divergent sequence.

The contraction mapping theorem states that if $f : \mathbb{R} \to \mathbb{R}$ (or $f : [a, \infty) \to [a, \infty]$) is a map such that for some $k$ in $(0, 1)$ and all $x$ and $y$ in $\mathbb{R}$, $|f(x) - f(y)| \leq k|x - y|$, then the iterates $f^n = f \circ \ldots \circ f$ ($n$ terms) of $f$ converge to a (unique) fixed point $\xi$ of $f$. This theorem can be accompanied by an example to show that the inequality cannot be replaced by the weaker condition $|f(x) - f(y)| < |x - y|$. The most common example of this type is $f(x) = x + 1/x$ acting on $[1, \infty)$. Then $f(x) > x$, so that $f$ has no fixed points. Also, for every $x$, the sequence $x, f(x), f^2(x), \ldots$ is strictly increasing and so must converge in the space $[-\infty, +\infty]$. In fact, $f^n(x) \to +\infty$, for otherwise $f^n(x) \to a$ for some real $a$, and then $f(f^n(x)) \to f(a)$ (because $f$ is continuous) so that $f(a) = a$, which is not so. Thus we define $f(+\infty)$ to be $+\infty$ and deduce that this example is no longer a counterexample. The following property clarifies these ideas and provides an elementary, but interesting, adjunct to the contraction mapping theorem. We just point out that a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| < |x - y|$ for all $x \neq y$ is called a contractive function.

We say that a function $f : [a, b] \to \mathbb{R}$ satisfies the Lipschitz condition with constant $L > 0$ if for all $x$ and $y$ in $[a, b]$, $|f(x) - f(y)| \leq L|x - y|$. A function that satisfies a Lipschitz condition is clearly continuous. Geometrically, if $f : [a, b] \to \mathbb{R}$ satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b],$$

then for any $x, y \in [a, b], x \neq y$, the inequality

$$\frac{|f(x) - f(y)|}{|x - y|} \leq L$$

indicates that the slope of the chord joining the points $(x, f(x))$ and $(y, f(y))$ on the graph of $f$ is bounded by $L$.

Using the fact that the real line is totally ordered, the following more general theorem with much more elementary proof is possible.

**Proposition 6.** Let $f : [a, b] \to [a, b]$ be a function that satisfies a Lipschitz condition with constant $L$. Let $x_1$ in $[a, b]$ be arbitrary and define $x_{n+1} = (1 - \lambda)x_n + \lambda f(x_n)$ where $\lambda = 1/(L + 1)$. If $(x_n)_{n\geq1}$ denotes the resulting
sequence then \((x_n)_{n \geq 1}\) converges monotonically to a point \(z\) in \([a, b]\) where \(f(z) = z\).

**Proof.** Without loss of generality we can assume \(f(x_n) \neq x_n\) for all \(n\). Suppose \(f(x_1) > x_1\) and let \(p\) be the first point greater than \(x_1\) such that \(f(p) = p\). Since \(f(x_1) > x_1\) and \(f(b) \leq b\), the continuity of \(f\) implies there is such a point.

Next, we prove the following claim. If \(x_1 < x_2 < \ldots < x_n < p\) and \(f(x_i) > x_i\) for \(i = 1, 2, \ldots, n\), then \(f(x_{n+1}) > x_{n+1}\) and \(x_{n+1} < p\). Indeed, suppose \(p < x_{n+1}\), then \(x_n < p < x_{n+1}\), hence \(0 < p - x_n < x_{n+1} - x_n = \lambda(f(x_n) - x_n)\). Therefore

\[
0 < \frac{1}{\lambda} |x_n - p| = (L + 1) |x_n - p| < |f(x_n) - x_n| \leq |f(x_n) - f(p)| + |p - x_n|.
\]

It follows that

\[
L|x_n - p| < |f(x_n) - f(p)|,
\]

which contradicts the fact that \(f\) is a Lipschitz function. Thus \(x_{n+1} < p\) and \(f(x_{n+1}) > x_{n+1}\) by the choice of \(p\), and the claim is proved.

Using the induction hypothesis it follows that \(x_n < x_{n+1} < p\) for all integers \(n\). Since a bounded monotonic sequence converges, \((x_n)_{n \geq 1}\) converges to some point \(z\). By the triangle inequality it follows that

\[
|z - f(z)| \leq |z - x_n| + |x_n - f(x_n)| + |f(x_n) - f(z)| = |z - x_n| + \frac{1}{\lambda} |x_{n+1} - x_n| + |f(x_n) - f(z)|.
\]

Since the right-hand side tends to 0 as \(n \to \infty\), we conclude that \(f(z) = z\). If \(f(x_1) < x_1\) a similar argument holds.

Applying a somewhat more sophisticated argument, one can allow \(\lambda\) to be any number less than \(2/(L + 1)\) but the resulting sequence \((x_n)_{n \geq 1}\) need not converge monotonically. The following example shows this last result is best possible.

Let \(f : [0, 1] \to [0, 1]\) be defined by

\[
f(x) = \begin{cases} 
  1, & 0 \leq x < \frac{L - 1}{2L} \\
  -Lx + \frac{1}{2}(L + 1), & \frac{L - 1}{2L} \leq x \leq \frac{L + 1}{2L} \\
  0, & \frac{L + 1}{2L} < x \leq 1,
\end{cases}
\]

where \(L > 1\) is arbitrary. Note that \(f\) satisfies a Lipschitz condition with constant \(L\). Let \(\lambda = 2/(L + 1)\) and let \(x_1 = (L - 1)/2L\). Then \(x_2 = (1 - \lambda)x_1 + \lambda f(x_1) = (L + 1)/2L\), \(x_3 = (1 - \lambda)x_2 + \lambda f(x_2) = (L - 1)/2L\), etc.
5. Picard sequences versus the cobweb model and qualitative analysis of markets

We start with the following elementary geometric interpretation of the Picard method. First, take a point \( A(x_1, f(x_1)) \) on the curve \( y = f(x) \). Next, consider the point \( B_1(f(x_1), f(x_1)) \) on the diagonal line \( y = x \) and then, project the point \( B_1 \) vertically onto the curve \( y = f(x) \) to obtain a point \( A_2(x_2, f(x_2)) \). Again, project \( A_2 \) horizontally to \( B_2 \) on \( y = x \) and then, project \( B_2 \) vertically onto \( y = f(x) \) to obtain \( A_3(x_3, f(x_3)) \). This process can be repeated a number of times. Often, it will weave a cobweb in which the fixed point of \( f \), that is, the point of intersection of the curve \( y = f(x) \) and the diagonal line \( y = x \), gets trapped. In fact, such trapping occurs if the slopes of tangents to the curve \( y = f(x) \) are smaller (in absolute value) than the slope of the diagonal line \( y = x \). The situation described above is illustrated in Figure 4.

![Figure 4](#) Picard sequence converging to a fixed point

When the slope condition is not met, then the points \( A_1, A_2 \ldots \) may move away from a fixed point. This case is depicted in Figure 5.

![Figure 5](#) Picard sequence diverging away from a fixed point
In mathematical economics, the behaviour described in Figure 4 corresponds to the convergence to an equilibrium point, while the framework described in Figure 5 describes the divergence from equilibrium.

A sufficient condition for the convergence of a Picard sequence, which is a formal analogue of the geometric condition of slopes mentioned above, is stated in the following result, which is also referred as the Picard convergence theorem.

**Theorem 5.** Let \( f : [a, b] \to [a, b] \) be a continuous function which is differentiable on \((a, b)\), with \(|f'(x)| < 1\) for all \(x \in (a, b)\).

Then \( f \) has a unique fixed point. Moreover, any Picard sequence for \( f \) is convergent and converges to the unique fixed point of \( f \).

**Proof.** We first observe that the Brouwer fixed point theorem implies that \( f \) has at least one fixed point. Next, assuming that \( f \) has two fixed points \( x_* \) and \( x^* \), the Lagrange mean value theorem implies that there exists \( \xi \in (a, b) \) such that

\[
|x_* - x^*| = |f(x_*) - f(x^*)| = |f'(|\xi|)| \cdot |x_* - x^*| < |x_* - x^*|
\]

a contradiction. Thus, \( f \) has a unique fixed point.

We point out that the condition \(|f'(x)| < 1\) for all \(x \in (a, b)\) is essential for the uniqueness of a fixed point. For example, if \( f : [a, b] \to [a, b] \) is defined by \( f(x) = x \), then \( f'(x) = 1 \) for all \( x \in [a, b] \) and every point of \([a, b]\) is a fixed point of \( f \).

We prove in what follows that the corresponding Picard sequence converges. Let \( x^* \) denote the unique fixed point of \( f \). Consider arbitrarily \( x_1 \in [a, b] \) and let \( (x_n)_{n=1} \subset [a, b] \) be the Picard sequence for \( f \) with its initial point \( x_1 \). This means that \( x_n = f(x_{n-1}) \) for all \( n \geq 2 \). Fix an integer \( n > 1 \). Thus, by the Lagrange mean value theorem, there exists \( \xi_n \) between \( x_n \) and \( x^* \) such that

\[
x_{n+1} - x^* = f(x_n) - f(x^*) = f'(\xi_n)(x_n - x^*)
\]

This implies that \(|x_{n+1} - x^*| < |x_n - x^*|\). Next, we prove that \( x_n \to x^* \) as \( n \to \infty \). Since \( (x_n)_{n=1} \) is bounded, it suffices to show that every convergent subsequence of \( (x_n)_{n=1} \) converges to \( x^* \). Let \( x \in \mathbb{R} \) and \( (x_{n_k})_{k=1} \) be a subsequence of \( (x_n)_{n=1} \) converging to \( x \). Then

\[
|x_{n_k} - x^*| \leq |x_{n_k+1} - x^*| \leq |x_n - x^*|.
\]

But \(|x_{n_k+1} - x^*| \to |x - x^*|\) and

\[
|x_{n_k+1} - x^*| = |f(x_{n_k}) - f(x^*)| \to |f(x) - f(x^*)|, \quad \text{as } k \to \infty.
\]

It follows that \(|f(x) - f(x^*)| = |x - x^*|\). Now, if \( x \neq x^* \), then by the Lagrange mean value theorem, there exists \( \xi \in (a, b) \) such that

\[
|x - x^*| = |f(x) - f(x^*)| = |f'(\xi)| \cdot |x - x^*| < |x - x^*|,
\]

\[
\text{where } |f'(\xi)| < 1.
\]
which is a contradiction. This proves that \( x \neq x^* \). \( \square \)

**Remark.** If the condition \( |f'(x)| < 1 \) for all \( x \in (a, b) \) is not satisfied, then \( f \) can still have a unique fixed point \( x^* \) but the Picard sequence \( (x_n)_{n \geq 1} \) with initial point \( x_1 \neq x^* \) may not converge to \( x^* \).

**Example.** Let \( f : [-1, 1] \to [-1, 1], f(x) = -x \). Then \( f \) is differentiable, \( |f'(x)| = 1 \) for all \( x \in [-1, 1] \), and \( x^* = 0 \) is the unique fixed point of \( f \). If \( x_1 \neq 0 \) then the corresponding Picard sequence is \( x_1, -x_1, x_1, -x_1, \ldots \), which oscillates between \( x_1 \) and \(-x_1\) and never reaches the fixed point. In geometric terms, the cobweb that we hope to weave just traces out a square over and over again.

When the hypotheses of the Picard convergence theorem are satisfied, a Picard sequence for \( f : [a, b] \to [a, b] \) with arbitrary \( x_1 \in [a, b] \) as its initial point, converges to a fixed point of \( f \). It is natural to expect that if \( x_1 \) is closer to the fixed point, then the convergence rate will be better. A fixed point of \( f \) lies not only in the range of \( f \) but also in the ranges of the iterates \( f \circ f, f \circ f \circ f, \) and so on. Thus, if \( \mathcal{R}_n \) is the range of the \( n \)-fold composite \( f \circ \cdots \circ f \) \((n \text{ times})\), then a fixed point is in each \( \mathcal{R}_n \). If only a single point belongs to \( \bigcap_{n=1}^\infty \mathcal{R}_n \), then we have found our fixed point. In fact, the Picard method amounts to starting with any \( x_1 \in [a, b] \) and considering the image of \( x_1 \) under the \( n \)-fold composite \( f \circ \cdots \circ f \).

**Example.** If \( f : [0, 1] \to [0, 1] \) is defined by \( f(x) = \frac{x+1}{3} \), then the \( n \)th iterate of \( f \) is given by

\[
f \circ \cdots \circ f(n \text{ times})(x) = \frac{3^x + 4^n - 1}{3 \cdot 4^n}
\]

and

\[
\mathcal{R}_n = \left[ \frac{1}{3} \left( 1 - \frac{1}{4^n} \right), \frac{1}{3} \left( 1 + \frac{2}{4^n} \right) \right].
\]

Thus, \( \bigcap_{n=1}^\infty \mathcal{R}_n = \{ \frac{1}{3} \} \), hence \( \frac{1}{3} \) is the unique fixed point of \( f \). In general, it is not convenient to determine the ranges \( \mathcal{R}_n \) for all \( n \). So, it is simpler to use the Picard method, but this tool will be more effective if the above observations are used to some extent in choosing the initial point.

The Picard convergence theorem was extended in [5] to a framework arising frequently in mathematical economics. This corresponds to the cobweb model that concerns a qualitative analysis of markets in which supply adjustments have a time lag and demand adjustments occur with no delay. We briefly describe what follows the cobweb model and we conclude with the cobweb theorem, which is a generalization of Theorem 5. Let \( s(p) \) denote the total quantity of the product that sellers are willing to supply at a given price level \( p > 0 \). Assume the demand function \( d(p) \) represent the total quantity of the product that buyers are willing to purchase at a given price
level $p$. The situation described in the next result corresponds to price converging to an equilibrium price and it is described by economists as a "stable equilibrium". This means that if a small extraneous disturbance occurs in the market, eventually price will again converges to some equilibrium price. The same result shows that disturbance should be be large enough to remove the equilibrium. Such a disturbance might be a depression, drought, or large recession.

**Theorem 6.** (Cobweb Theorem) Let $s$ and $d$ be real-valued functions of the real variable $p > 0$, and suppose that the graphs of $s$ and $d$ intersect at the point $(p^*, q^*)$ where $q^* > 0$. Let $I$ be a closed interval centered at $p^*$ on which functions $s$ and $d$ have nonvanishing continuous derivatives. Define sequences $(p_n)$ and $(q_n)$ by letting $p_0$ be any element of $I$, $q_n = s(p_{n-1})$ and $p_n = d^{-1}(q_n)$ for all $n \geq 1$. Assume that $|s'(p)| < |d'(p)|$ for all $p$ in $I$. Then $\lim_{n \to \infty} p_n = p^*$ and $\lim_{n \to \infty} q_n = q^*$.

The proof of Theorem 6 relies on the Cauchy mean value theorem; we refer to [5] for details and related properties.

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**References**


