SOLUTIONS WITH BOUNDARY BLOW-UP FOR A CLASS OF NONLINEAR ELLIPTIC PROBLEMS

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Abstract. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \). We consider the logistic equation \( \Delta u + au = b(x)f(u) \) in \( \Omega \), where \( a \) is a real number, \( b \) is continuous, \( b \geq 0, b \neq 0 \), and \( f \in C^1 \) is a positive function satisfying the Keller–Osserman condition and such that \( f(u)/u \) is increasing on \( (0, \infty) \). We prove that a necessary and sufficient condition for the existence of a positive solution blowing-up at the boundary of \( \Omega \) is that \( a \in (-\infty, \lambda_{\infty,1}) \), where \( \lambda_{\infty,1} \) is the first eigenvalue of \( (-\Delta) \) in \( H_0^1(\Omega_0) \) and \( \Omega_0 = \text{int} \{ x \in \Omega; b(x) = 0 \} \). Our framework includes the case when the potential \( b \) vanishes at some points on \( \partial \Omega \) or even on the whole boundary.

1. The Main Result

This paper originated with the recent work Alama–Tarantello [1] which contains an exhaustive study of the logistic problem

\[
\begin{aligned}
\Delta u + \lambda u = b(x)f(u) & \quad \text{in } \Omega, \\
u = 0 & \quad \text{on } \partial \Omega, \\
u > 0 & \quad \text{in } \Omega,
\end{aligned}
\]

(1)

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) \((N \geq 2)\), \( \lambda \) is a real parameter and \( b \in C^{0,\alpha}(\Omega) \), \( 0 < \alpha < 1 \) satisfies \( b \geq 0 \) and \( b \neq 0 \) in \( \Omega \). It is worth pointing out here that if \( f(u) = u^{(N+2)/(N-2)} \) \((N \geq 3)\), then this equation originates from

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the Yamabe problem, which is a basic problem in Riemannian geometry (see, e.g., [9]).

The zero set
$$\Omega_0 := \text{int} \{ x \in \Omega : b(x) = 0 \}$$
plays an important role in the understanding of this problem. We shall assume throughout that \( \Omega_0 \subset \Omega \) and \( b > 0 \) in \( \Omega \setminus \Omega_0 \).

Suppose that \( f \in C^1[0, \infty) \) satisfies
\[
(f_1) \quad f \geq 0 \text{ and } f(u)/u \text{ is increasing on } (0, \infty).
\]

Following Alama-Tarantello [1], define by \( H_\infty \) the Dirichlet Laplacian on the set \( \Omega_0 \subset \Omega \) as the unique self-adjoint operator associated to the quadratic form
\[
\xi(u) = \int_\Omega |\nabla u|^2 \, dx \text{ with form domain } H^1_D(\Omega_0) = \{ u \in H^1_0(\Omega) : u(x) = 0 \text{ for a.e. } x \in \Omega \setminus \Omega_0 \}.
\]

If \( \partial \Omega_0 \) satisfies an exterior cone condition, then \( H^1_D(\Omega_0) \) coincides with \( H^1_0(\Omega_0) \) and \( H_\infty \) is the classical Laplace operator with Dirichlet condition on \( \partial \Omega_0 \) (see [1]).

Let \( \lambda_{\infty,1} \) be the first Dirichlet eigenvalue of \( H_\infty \) in \( \Omega_0 \). We understand \( \lambda_{\infty,1} = +\infty \) if \( \Omega_0 = \emptyset \).

Set \( \mu_0 := \lim_{u \searrow 0} \frac{f(u)}{u} \) and \( \mu_\infty := \lim_{u \to \infty} \frac{f(u)}{u} \). The results of Alama and Tarantello rely on the existence of a principal eigenvalue for the operator \(-\Delta + \mu b\) in the limiting cases \( \mu = \mu_0 \) and \( \mu = \mu_\infty \). Denote by \( \lambda_1(\mu_0) \) (resp., \( \lambda_1(\mu_\infty) \)) the first eigenvalue of \( H_{\mu_0} = -\Delta + \mu_0 b \) (resp., \( H_{\mu_\infty} = -\Delta + \mu_\infty b \)) in \( H^1_0(\Omega) \). Recall that \( \lambda_1(+\infty) = \lambda_{\infty,1} \).

The main result of [1] (see also [6], [16]) asserts that problem (1) has a solution \( u_\lambda \) if and only if \( \lambda \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty)) \), and, moreover, \( u_\lambda \) is the unique solution of (1) (see [1, Theorem A (bis)]). We point out that neither assumption on the smoothness of \( \partial \Omega_0 \) nor topological restriction on \( \Omega \) are made in [1].

Our purpose is to give a corresponding necessary and sufficient condition, but for solutions of the problem
\[
\begin{cases}
\Delta u + au = b(x)f(u) & \text{in } \Omega, \\
\lim_{\text{dist}(x,\partial\Omega) \to 0} u(x) = +\infty, \\
u \geq 0 & \text{in } \Omega,
\end{cases}
\]
where \( a \) is a real parameter. A solution of (2) is called large (or explosive) solution.

There is a vast literature on nonlinear elliptic problems having solutions that blow-up at the boundary, starting with the pioneering papers [14], [8], [13], [10]. We also refer to the paper [15], where there are studied large solutions of the problem
\[
\Delta u = b(x)u^{(N+2)/(N-2)}
\]
in a ball, in particular for questions of existence, uniqueness and boundary behaviour.

We impose the natural Keller–Osserman condition

\[ (f_2) \quad \int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where} \quad F(t) = \int_0^t f(s) \, ds. \]

We recall (see [8, 13]) that this condition is necessary and sufficient for the existence of a large solution to problem \( \Delta u = h(u) \), where \( h \in C^1 \), \( h(0) = 0 \), \( h' \geq 0 \) and \( h > 0 \) on \((0, \infty)\).

Examples of non-linearities satisfying \((f_1)\) and \((f_2)\): (i) \( f(u) = e^u - 1 \); (ii) \( f(u) = u^p, p > 1 \); (iii) \( f(u) = u [\ln(u + 1)]^p, p > 2 \).

**Remark 1.** We have \( \mu_\infty := \lim_{u \to \infty} f(u)/u = \lim_{u \to \infty} f'(u) = \infty \). Indeed, by l’Hospital’s rule, we have \( \lim_{u \to \infty} F(u)/u^2 = \mu_\infty/2 \). But, by \((f_2)\), we deduce that \( \mu_\infty = \infty \). Then, by \((f_1)\) we find that \( f'(u) \geq f(u)/u \) for any \( u > 0 \), which shows that \( \lim_{u \to \infty} f'(u) = \infty \).

Our main result is

**Theorem 1.1.** Assume conditions \((f_1)\) and \((f_2)\) hold. Then problem (2) has a solution if and only if \( a \in (-\infty, \lambda_\infty, 1) \). Moreover, in this case, the solution is positive.

We point out that our framework in the above result includes the case when \( b \) vanishes at some points on \( \partial \Omega \), or even if \( b \equiv 0 \) on \( \partial \Omega \). In this sense, our result responds to a question raised to one of us by Professor Haim Brezis in Paris, May 2001.

Our result also applies to problems on Riemannian manifolds if \( \Delta \) is replaced by the Laplace–Beltrami operator

\[ \Delta_B = \frac{1}{\sqrt{c}} \frac{\partial}{\partial x_i} \left( \sqrt{c} a_{ij}(x) \frac{\partial}{\partial x_i} \right), \quad c := \det(a_{ij}), \]

with respect to the metric \( ds^2 = c_{ij} \, dx_i \, dx_j \), where \((c_{ij})\) is the inverse of \((a_{ij})\). In this case our result applies to concrete problems arising in Riemannian geometry. For instance, (cf. Loewner-Nirenberg [10]) if \( \Omega \) is replaced by the standard \( N \)-sphere \((S^N, g_0)\), \( \Delta \) is the Laplace-Beltrami operator \( \Delta_{g_0}, a = N(N - 2)/4 \), and \( f(u) = (N - 2)/(4(N - 1)) \, u^{(N+2)/(N-2)} \), we find the prescribing scalar curvature equation on \( S^N \).
2. An auxiliary comparison principle

Lemma 1. Let $\omega \subset \mathbb{R}^N$ be a smooth bounded domain. Assume $f$ is continuous on $(0, \infty)$, $f(u)/u$ is increasing on $(0, \infty)$, and $p, q, r$ are $C^0,\mu$-functions on $\omega$ such that $r \geq 0$ and $p > 0$ in $\omega$. Let $u_1, u_2 \in C^2(\omega)$ be positive functions such that

\[ \Delta u_1 + q(x)u_1 - p(x)f(u_1) + r(x) \leq \Delta u_2 + q(x)u_2 - p(x)f(u_2) + r(x) \quad \text{in} \ \omega \]

\[ \lim_{\text{dist}(x, \partial \omega) \to 0} (u_2 - u_1)(x) \leq 0. \]

Then $u_1 \geq u_2$ in $\omega$.

Proof. We use the same method as in the proof of Lemma 1.1 in Marcus-Véron [12] (see also [7, Lemma 2.1]), that goes back to Benguria-Brezis-Lieb [2].

By (3) we obtain, for any non-negative function $\phi \in H^1(\omega)$ with compact support in $\omega$,

\[ \int_\omega (\nabla u_1 \cdot \nabla \phi - qu_1 \phi + pf(u_1)\phi - r\phi) \geq 0 \geq \int_\omega (\nabla u_2 \cdot \nabla \phi - qu_2 \phi + pf(u_2)\phi - r\phi). \]

Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

\[ \omega(\varepsilon_1, \varepsilon_2) = \{ x \in \omega : u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1 \} \]

\[ v_i = (u_i + \varepsilon_i)^{-1}((u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2)^+, \quad i = 1, 2. \]

Notice that $v_i \in H^1(\omega)$ and, in view of (4), it has compact support in $\omega$. Using (5) with $\phi = v_i$ and taking into account that $v_i$ vanishes outside $\omega(\varepsilon_1, \varepsilon_2)$ we find

\[ \int_{\omega(\varepsilon_1, \varepsilon_2)} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1) \, dx \geq \int_{\omega(\varepsilon_1, \varepsilon_2)} p(f(u_2)v_2 - f(u_1)v_1) \, dx \]

\[ + \int_{\omega(\varepsilon_1, \varepsilon_2)} q(u_1v_1 - u_2v_2) \, dx + \int_{\omega(\varepsilon_1, \varepsilon_2)} r(v_1 - v_2) \, dx. \]

A simple computation shows that the integral in the left-hand side of (6) equals

\[ -\int_{\omega(\varepsilon_1, \varepsilon_2)} \left( \left| \nabla u_1 - \frac{u_2 + \varepsilon_2}{u_1 + \varepsilon_1} \nabla u_1 \right|^2 + \left| \nabla u_1 - \frac{u_1 + \varepsilon_1}{u_2 + \varepsilon_2} \nabla u_2 \right|^2 \right) \, dx \leq 0. \]

Passing to the limit as $0 < \varepsilon_2 < \varepsilon_1 \to 0$, the first term in the right-hand side converges to

\[ \int_{\omega(0, 0)} p \left( \frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1} \right)(u_2^2 - u_1^2) \, dx, \]
the second term goes to 0, while the third one converges to
\[ \int_{\omega(0,0)} \frac{r(u_2 - u_1)^2(u_2 + u_1)}{u_1u_2} \, dx \geq 0. \]
Hence we avoid a contradiction only in the case when \( \omega(0,0) \) has measure 0, which means that \( u_1 \geq u_2 \) on \( \omega \).

3. Proof of Theorem 1.1

A. Necessary condition. Let \( u_{\infty} \) be a large solution of problem (2). We claim that \( u_{\infty} \) is positive. Indeed, since \( u_{\infty}(x) \to \infty \) as \( \text{dist} (x, \partial \Omega) \to 0 \), there exists a smooth open set \( \omega \subset \subset \Omega \) such that \( u_{\infty} > 0 \) in \( \Omega \setminus \omega \). So, it is enough to show that \( u_{\infty} > 0 \) in \( \omega \). For this aim, set \( M_0 := 1 + \sup_{\omega} b > 0 \) and consider the problem

(7)
\[
\begin{cases}
\Delta u = |a|u + M_0 f(u) & \text{in } \omega, \\
u = u_{\infty} & \text{on } \partial \omega, \\
u \geq 0 & \text{in } \omega.
\end{cases}
\]

By Proposition 2.1 in [11] (see also [5, Theorem 5]), this problem has a unique solution \( u_0 \) and, moreover, \( u_0 > 0 \) in \( \omega \) and \( u_{\infty} \) is supersolution for problem (7), so \( u_{\infty} \geq u_0 > 0 \) in \( \omega \) and our claim is proved.

Suppose \( \lambda_{\infty,1} \) is finite. Arguing by contradiction, let us assume \( a \geq \lambda_{\infty,1} \). Set \( \lambda \in (\lambda_1(\mu_0), \lambda_{\infty,1}) \) and denote by \( u_\lambda \) the unique positive solution of problem (1).

We have
\[
\begin{cases}
\Delta(Mu_{\infty}) + \lambda_{\infty,1}(Mu_{\infty}) \leq b(x)f(Mu_{\infty}) & \text{in } \Omega, \\
Mu_{\infty} = \infty & \text{on } \partial \Omega, \\
Mu_{\infty} \geq u_\lambda & \text{in } \Omega,
\end{cases}
\]
where \( M := \max \{ \max_{\Omega} u_\lambda / \min_{\Omega} u_{\infty}; 1 \} \). By the sub-super solutions method we conclude that problem (1) with \( \lambda = \lambda_{\infty,1} \) has at least a positive solution (between \( u_\lambda \) and \( Mu_{\infty} \)). But this is a contradiction. So, necessarily, \( a \in (-\infty, \lambda_{\infty,1}) \).

B. Sufficient condition. This will be proved with the aid of several results. From now on we assume throughout the paper that \( f \) satisfies \((f_1)\) and \((f_2)\).

Lemma 2. Let \( \omega \) be a smooth bounded domain in \( \mathbb{R}^N \). Assume \( p, q, r \) are \( C^{0,\mu} \)-functions on \( \overline{\omega} \) such that \( r \geq 0 \) and \( p > 0 \) in \( \overline{\omega} \). Then for any non-negative function \( 0 \neq \Phi \in C^{0,\mu}(\partial \omega) \) the boundary value problem

(8)
\[
\begin{cases}
\Delta u + q(x)u = p(x)f(u) - r(x) & \text{in } \omega, \\
u > 0 & \text{in } \omega, \\
u = \Phi & \text{on } \partial \omega,
\end{cases}
\]
has a unique solution.
Proof. By Lemma 1, problem (8) has at most one solution. The existence of a positive solution will be obtained by device of sub and super-solutions.

Set \( m := \inf_{\omega} p > 0 \). Define \( \bar{f}(u) = mf(u) - \|q\|_\infty u - \bar{r} \), where \( \bar{r} := \sup_{\omega} r + 1 > 0 \). Let \( t_0 \) be the unique positive solution of the equation \( \bar{f}(u) = 0 \). By Remark 1 we derive that \( \lim_{u \to \infty} \frac{f(u)}{u} = m > 0 \). Combining this with \( (f_2) \), we conclude that the function \( \varphi(w) = \bar{f}(w + t_0) \) defined for \( w \geq 0 \) satisfies the assumptions of Theorem III in [8]. It follows that there exists a positive large solution for the equation \( \Delta w = \varphi(w) \) in \( \omega \). Thus the function \( \bar{u}(x) = w(x) + t_0 \), for all \( x \in \omega \), is a positive large solution of the problem

\[(9) \quad \Delta u + \|q\|_\infty u = mf(u) - \bar{r} \quad \text{in } \omega.\]

By Proposition 2.1 in [11], the boundary value problem

\[
\begin{cases}
\Delta u = \|q\|_\infty u + \|p\|_\infty f(u) & \text{in } \omega, \\
u > 0 & \text{in } \omega, \\
u = \Phi & \text{on } \partial \omega,
\end{cases}
\]

has a unique classical solution \( u \). By Lemma 1, we find that \( u \leq \bar{u} \) in \( \omega \) and \( u \) (resp., \( \bar{u} \)) is a positive sub-solution (resp., super-solution) of problem (8). It follows that (8) has a unique solution. \( \square \)

Under the assumptions of Lemma 2 we obtain the following result which generalizes Lemma 1.3 in [12].

**Corollary 1.** There exists a positive large solution of the problem

\[(10) \quad \Delta u + q(x)u = p(x)f(u) - r(x) \quad \text{in } \omega.\]

**Proof.** Set \( \Phi = n \) and let \( u_n \) be the unique solution of (8). By Lemma 1, \( u_n \leq u_{n+1} \leq \bar{u} \) in \( \omega \), where \( \bar{u} \) denotes a large solution of (9). Thus \( \lim_{n \to \infty} u_n(x) = u_\infty(x) \) exists and is a positive large solution of (10). Furthermore, every positive large solution of (10) dominates \( u_\infty \), i.e., the solution \( u_\infty \) is the minimal large solution. This follows from the definition of \( u_\infty \) and Lemma 1. \( \square \)

**Lemma 3.** If \( 0 \neq \Phi \in C^0(\bar{\Omega}) \) is a non-negative function and \( b > 0 \) on \( \partial \Omega \), then the boundary value problem

\[(11) \quad \begin{cases}
\Delta u + au = b(x)f(u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = \Phi & \text{on } \partial \Omega,
\end{cases}\]

has a solution if and only if \( a \in (-\infty, \lambda_\infty) \). Moreover, in this case, the solution is unique.
Proof. The first part follows with the same arguments as in the proof of Theorem 1.1 (necessary condition).

For the sufficient condition, fix $a < \lambda_{\infty,1}$ and let $\lambda_{\infty,1} > \lambda > \max \left\{ a, \lambda_1(\mu_0) \right\}$. Let $u_*$ be the unique positive solution of (1) with $\lambda = \lambda_*$.

Let $\Omega_i (i = 1, 2)$ be subdomains of $\Omega$ such that $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega_2 \subset \subset \Omega$ and $\Omega \setminus \overline{\Omega_1}$ is smooth. We define $u_+ \in C^2(\Omega)$ as a positive function in $\Omega$ such that $u_+ \equiv u_\infty$ on $\Omega \setminus \Omega_2$ and $u_+ \equiv u_*$ on $\Omega_1$. Here $u_\infty$ denotes a positive large solution of (10) for $p(x) = b(x)$, $r(x) = 0$, $q(x) = a$ and $\omega = \Omega \setminus \overline{\Omega_1}$. Using Remark 1 and the fact that $b_0 := \inf_{\Omega_2 \setminus \Omega_1} b > 0$, it is easy to check that if $C > 0$ is large enough then $v_\Phi = Cu_+$ satisfies

$$
\begin{cases}
\Delta v_\Phi + a v_\Phi \leq b(x) f(v_\Phi) & \text{in } \Omega, \\
v_\Phi = \infty & \text{on } \partial \Omega, \\
v_\Phi \geq \max_{\partial \Omega} \Phi & \text{in } \Omega.
\end{cases}
$$

By Proposition 2.1 in [11], there exists a unique classical solution $v_\Phi$ of the problem

$$
\begin{cases}
\Delta v_\Phi = |a| v_\Phi + \|b\|_{\infty} f(v_\Phi) & \text{in } \Omega, \\
v_\Phi > 0 & \text{in } \Omega, \\
v_\Phi = \Phi & \text{on } \partial \Omega.
\end{cases}
$$

It is clear that $v_\Phi$ is a positive sub-solution of (11) and $v_\Phi \leq \max_{\partial \Omega} \Phi \leq v_\Phi$ in $\Omega$. Therefore, by the sub-super solution method, problem (11) has at least a solution $v_\Phi$ between $v_\Phi$ and $\overline{v_\Phi}$. Next, the uniqueness of solution to (11) can be obtained by using essentially the same technique as in [4, Theorem 1] or [3, Appendix II].

Proof of Theorem 1.1 completed. Fix $a \in (-\infty, \lambda_{\infty,1})$. Two cases may occur:

Case 1: $b > 0$ on $\partial \Omega$. Denote by $v_n$ the unique solution of (11) with $\Phi \equiv n$. For $\Phi \equiv 1$, set $v := v_\Phi$ and $V := \overline{v_\Phi}$, where $v_\Phi$ and $\overline{v_\Phi}$ are defined in the proof of Lemma 3. The sub and super-solutions method combined with the uniqueness of solution of (11) shows that $v \leq v_n \leq v_{n+1} \leq V$ in $\Omega$. Hence $v_\infty(x) := \lim_{n \to \infty} v_n(x)$ exists and is a solution of (2).

Case 2: $b \geq 0$ on $\partial \Omega$. Let $z_n$ ($n \geq 1$) be the unique solution of (8) for $p \equiv b+1/n$, $r \equiv 0$, $q \equiv a$, $\Phi \equiv n$ and $\omega = \Omega$. By Lemma 1, $(z_n)$ is non-decreasing. Moreover, $(z_n)$ is uniformly bounded on every compact subdomain of $\Omega$. Indeed, if $K \subset \Omega$ is an arbitrary compact set, then $d := \text{dist}(K, \partial \Omega) > 0$. Choose $\delta \in (0, d)$ small enough so that $\overline{\Omega_0} \subset C_\delta$, where $C_\delta = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \delta\}$. Since
b > 0 on ∂Cδ. Case 1 allows us to define z_+ as a solution of (2) for Ω = Cδ. Using Lemma 1 for p ≡ b + 1/n, r ≡ 0, q ≡ a and ω = Cδ we obtain z_n ≤ z_+ in Cδ for all n ≥ 1. So, (z_n) is uniformly bounded on K. By the monotonicity of (z_n), we conclude that z_n → z in L^∞(Ω). Finally, standard elliptic regularity arguments lead to z_n → z in C^{2,α}_{loc}(Ω). This completes the proof of Theorem 1.1.

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