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DOUBLE PHASE OBSTACLE PROBLEMS WITH VARIABLE EXPONENT

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Abstract. This paper is devoted to the study of a quasilinear elliptic inclusion problem driven by a double phase differential operator with variable exponents, an obstacle effect and a multivalued reaction term with gradient dependence. By using an existence result for mixed variational inequalities with multivalued pseudomonotone operators and the theory of nonsmooth analysis, we examine the nonemptiness, boundedness and closedness of the solution set to the problem under consideration. In the second part of the paper, we present some convergence analysis for approximated problems. To be more precise, when the obstacle function is approximated by a suitable sequence, applying a generalized penalty technique, we introduce a family of perturbed problems without constraints associated to our problem and prove that the solution set of the original problem can be approached by the solution sets of the perturbed problems in the sense of Kuratowski.

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1. Introduction

Originally, the study of obstacle problems is due the pioneering contribution by Stefan [45] in which the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water, was studied. Such kind of problems also frequently occur in physics, biology, and financial mathematics. Some important examples are the dam problem, the Hele-Shaw flow, pricing of American options, quadrature domains and random matrices. Another groundbreaking work in this direction has been published by J.-L. Lions [31] who considered the following problem: find the equilibrium position $u = u(x)$ of an elastic membrane which lies above a given obstacle $\psi = \psi(x)$ with $x \in \Omega$ and $\Omega \subset \mathbb{R}^2$ being a bounded smooth domain. It turns out that the equilibrium position is the unique solution of the Dirichlet energy functional

$$\min_{v \in K} \int_\Omega |\nabla v|^2 \,\mathrm{d}x$$

with K being an appropriate convex set of functions greater or equal to the obstacle ψ. Such problem can be equivalently written as a variational inequality of the following form: find $u \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u)\,\mathrm{d}x \geq 0 \quad \text{for all } v \in K.$$

It is clear that the solution u solves the equation $\Delta u = 0$ in the region $[v > \psi]$ (so the membrane is above the obstacle) while in the other region the membrane is equal to the obstacle, that is, $u = \psi$. Usually, the region $[v = \psi]$ is called the contact set and the interface that separates the two regions is the free boundary. Different classes of obstacle problems appear naturally when describing phenomena in the real world. Several interesting models, such as fluid filtration through porous medium, osmosis, optimal stopping and heat control, are explained and studied in the monographs of Duvaut and Lions [17] and Rodrigues [43].

In the current paper, we are interested in the study of a quasilinear elliptic obstacle inclusion problem with a double phase differential operator with variable exponents and a multivalued convection term. To this end, given a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$, we consider the problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)& \\ +\beta|u|^{\theta(x)-2}u \in f(x,u,\nabla u) \text{ in } \Omega,& \end{aligned} \tag{1.1}$$

$$
\begin{aligned}
u &\leq \Phi && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

where $0 \leq \mu(\cdot) \in L^{\infty}(\Omega)$, $\beta > 0$, $p, q, \theta \colon \overline{\Omega} \to (1, +\infty)$ are continuous functions, $\Phi \colon \Omega \to [0, \infty)$ is a given obstacle function and $f \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ is multivalued convection term. Problem (1.1) appears naturally when considering optimal stopping problems for Lévy processes with jumps, which arise for example as option pricing models in mathematical finance.

Problem (1.1) includes several interesting special cases which are listed below.

(i) If $\beta = 0$, then problem (1.1) becomes

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &\in f(x, u, \nabla u) && \text{in } \Omega,\\
u &\leq \Phi && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.2}
$$

which has not been investigated yet.

(ii) If $\Phi \equiv +\infty$, then problem (1.1) turns into

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)&\\
+\beta|u|^{\theta(x)-2}u &\in f(x, u, \nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega,
\end{aligned}
\tag{1.3}
$$

which has not been studied yet.

(iii) If $\beta = 0$ and $\Phi \equiv +\infty$, then problem (1.1) reduces to the following elliptic inclusion problem without obstacle

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) &\in f(x, u, \nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
\tag{1.4}
$$

If f is a single valued operator, problem (1.4) has been recently studied by Crespo-Blanco, Gasiński, Harjulehto and Winkert [14].

(iv) When p, q and θ are constants, then problem (1.1) can be formulated by the following multivalued double phase obstacle problem

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right)&\\
+\beta|u|^{\theta-2}u &\in f(x, u, \nabla u) && \text{in } \Omega,\\
u &\leq \Phi && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
\tag{1.5}
$$

This problem has not been studied yet. If in addition $\beta = 0$, then (1.5) can be written as

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &\in f(x,u,\nabla u) && \text{in } \Omega,\\ u &\le \Phi && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

which has been considered by Zeng, Gasiński, Winkert and Bai [47, 49].

Therefore, the novelty of the current paper is the fact that several interesting and challenging phenomena are considered into one problem. To be more precise, problem (1.1) combines the following effects:

(i) a double phase differential operator with variable exponents which extends the isotropic case to the anisotropic one;
(ii) a multivalued convection term;
(iii) an obstacle restriction;
(iv) the functional framework on Musielak-Orlicz Sobolev spaces for variable exponents.

The main contribution of this paper is twofold. The first goal is to study the solution set $\mathcal{S}$ of problem (1.1) and it turns out that this set is nonempty (so an existence result), bounded and closed. Our method is based on the theory of nonsmooth analysis and an existence theorem for mixed variational inequalities with multivalued pseudomonotone operators. Since the obstacle effect leads to various difficulties in obtaining the exact and numerical solutions, some appropriate and useful approximating methods have been introduced and developed to overcome the obstacle constraints. Based on this, when the obstacle function is approximated by a suitable sequence, the second contribution of the paper is aimed to introduce a family of perturbed problems corresponding to problem (1.1) without constraints and to establish a critical convergence theorem which reveals that the solution set of the variable exponent double phase obstacle problem can be approximated by the solution sets of perturbed problems, denoted by $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$, in the sense of Kuratowski. More precisely, we establish the following relation

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S},$$

where w-$\limsup_{n\to\infty}\mathcal{S}_n$ is the weak Kuratowski upper limit of $\mathcal{S}_n$ and s-$\limsup_{n\to\infty}\mathcal{S}_n$ stands for the strong Kuratowski upper limit of $\mathcal{S}_n$. As far as we know this is the first work combining a variable exponent double phase operator with a multivalued convection term and an obstacle effect.

Originally, the double phase setting is due to Zhikov [52] who introduced and studied the integral functional

$$J(u) = \int \left(|\nabla u|^p + \mu(x)|\nabla u|^q\right) \mathrm{d}x \tag{1.6}$$

in order to describe models for strongly anisotropic materials. The functional $J(\cdot)$ is related to the differential operator

$$u \mapsto -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right). \tag{1.7}$$

Physically, the integral functional (1.6) illustrates the phenomenon that the energy density changes its ellipticity and growth properties according to the point in the domain. Mathematically, the behavior of $J(\cdot)$ is related to the sets on which the weight function $\mu(\cdot)$ vanishes or not. Therefore, we have two phases ($\mu(x) = 0$ or $\neq 0$) and so we call it double phase.

Based on the recent results of Crespo-Blanco, Gasiński, Harjulehto and Winkert [14], we extend the isotropic double phase operator in (1.7) to the following anisotropic one

$$u \mapsto -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right),$$

in which the exponents are now functions. We point out that isotropic and anisotropic double phase differential operators and related energy functionals describe several natural phenomena and model numerous problems in Mechanics, Physics and Engineering Sciences. In the elasticity theory, for example, the modulating coefficient $\mu(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents $q(x)$ and $p(x)$, see Zhikov [53]. In general, equations driven by the sum of two differential operators of different nature arise often in mathematical models of physical processes. We refer to the papers of Bahrouni, Rădulescu and Repovš [5] for transonic flow problems and of Cherfils and Il'yasov [10] for reaction diffusion systems.

As already mentioned there are only few works dealing with similar variable exponent double phase operators as in our work. In 2018, Zhang and Rădulescu [51] considered the problem

$$-\operatorname{div}\mathrm{A}(x,\nabla u) + V(x)|u|^{\alpha(x)-2}u = f(x,u) \tag{1.8}$$

where A fulfills certain $(p(x), q(x))$-growth conditions. Using a variational approach and critical point theory in Orlicz-Sobolev spaces with variable exponent, the existence of a pair of nontrivial constant sign solutions and infinitely many solutions of (1.8), respectively, was shown. Related results can be found in the work of Shi, Rădulescu, Repovš and Zhang [44]. Very recently, Bahrouni, Rădulescu and Winkert [7] obtained the existence of stationary waves under quite general assumptions based on pseudomonotone operators for Baouendi-Grushin type problems with convection given in the form

$$\begin{aligned} -\Delta_{G(x,y)}u + A(x,y)(|u|^{G(x,y)-1} + |u|^{G(x,y)-3})u &= f\left((x,y),u,\nabla u\right) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $G\colon \overline{\Omega} \to (1,\infty)$ is a continuous function and $\Delta_{G(x,y)}$ stands for the Baouendi-Grushin operator with variable coefficient. We also refer to the related work of Bahrouni, Rădulescu and Repovš [5, 6]. A first parabolic version of anisotropic double phase problems has been developed by Arora and Shmarev [4] (see also Arora [2] and Arora and Shmarev [3]) who studied the problem

$$u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + a(x)|\nabla u|^{q(x)-2}\nabla u\right) = F(x,u) \quad \text{in } Q_T = \Omega \times (0,T).$$

Under various conditions on the right-hand side the authors proved the existence of a unique strong solution with a certain kind of regularity. We also mention some papers dealing with existence results for $p(x)$- or $(p(x),q(x))$-Laplacian problems, see, for example, Cencelj, Rădulescu and Repovš [9], Gasiński and Papageorgiou [21], Vetro and Vetro [46] and the references therein.

Finally, we mention some existence and regularity results for isotropic double phase problems (or related operators) with different right-hand sides (single valued or multivalued and/or convection). We refer to the works of Alves, Garain and Rădulescu [1], Baroni, Colombo and Mingione [8], Colasuonno and Squassina [11], Colombo and Mingione [12, 13], De Filippis and Mingione [15], El Manouni, Marino and Winkert [18], Farkas, Fiscella and Winkert [19], Farkas and Winkert [20], Gasiński and Papageorgiou [22, 23], Gasiński and Winkert [24, 25, 26], Liu and Dai [32], Marino and Winkert [33] Papageorgiou, Rădulescu and Repovš [36, 37, 38], Papageorgiou, Vetro and Vetro [39], Zeng, Bai, Gasiński and Winkert [48], Zeng, Rădulescu and Winkert [50] and the references therein. We also mention the overview article of Rădulescu [41] about isotropic and anisotropic double phase problems and the recent article of Mingione and Rădulescu [35] concerning recent developments for problems with nonstandard growth and nonuniform ellipticity.

The paper is organized as follows. In Section 2, we will recall some necessary and useful preliminaries such as the Dirichlet eigenvalue problem for the r-Laplacian ($1 < r < \infty$), an existence theorem for mixed variational inequalities involving multivalued pseudomonotone operators, variable exponent Lebesgue and Sobolev spaces as well as Musielak-Orlicz spaces $L^{\mathcal{H}}(\Omega)$ and its corresponding Sobolev spaces $W_0^{1,\mathcal{H}}(\Omega)$, respectively. Section 3 is devoted to the properties of the solution set to problem (1.1) which is nonempty, bounded and closed, see Theorem 3.3. Finally, in Section 4, when the obstacle function is approximated by a suitable sequence, applying a generalized penalty technique, we are going to introduce a family of perturbed problems without constraints associated to our problem and prove that the solution set of (1.1) can be approached by the solution sets of the perturbed problems in the sense of Kuratowski.

2. Preliminaries

In this section, we present the main tools which are needed in the sequel. To this end, let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$ and let $1 \leq s < \infty$. We denote by $L^s(\Omega) := L^s(\Omega;\mathbb{R})$ and $L^s(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces endowed with the norm $\|\cdot\|_s$, that is,

$$\|u\|_s := \Big(\int_\Omega |u|^s \, \mathrm{d}x\Big)^{\frac{1}{s}} \quad \text{for all } u \in L^s(\Omega).$$

We set

$$L^s(\Omega)_+ := \{u \in L^s(\Omega) \,:\, u(x) \geq 0 \text{ for a.a. } x \in \Omega\}.$$

Moreover, $W^{1,s}(\Omega)$ stands for the Sobolev space endowed with the norm $\|\cdot\|_{1,s}$, namely,

$$\|u\|_{1,s} := \|u\|_s + \|\nabla u\|_s \quad \text{for all } u \in W^{1,s}(\Omega),$$

where $\|\nabla u\|_s = \|\,|\nabla u|\,\|_s$.

Let $s > 1$. We recall the well-known eigenvalue problem for the s-Laplacian with homogeneous Dirichlet boundary condition given by

$$\begin{aligned} -\Delta_s u &= \lambda |u|^{s-2} u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

It is well-known that the first eigenvalue, denoted by $\lambda_{1,s}$, is positive, simple and isolated. Moreover, it can be variationally characterized through

$$\lambda_{1,s} := \inf_{u \in W^{1,s}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_s}{\|u\|_s}$$

see Lê [30].

We now recall some basic properties of Lebesgue and Sobolev spaces with variable exponent; see Rădulescu and Repovš [42] for more details. We first introduce a subset $C_+(\overline{\Omega})$ of $C(\overline{\Omega})$ defined by

$$C_+(\overline{\Omega}) := \left\{ h \in C(\overline{\Omega}) \,:\, 1 < h(x) \text{ for all } x \in \overline{\Omega} \right\}$$

and denote by $M(\Omega)$ the space of all measurable functions $u\colon \Omega \to \mathbb{R}$. For any $r \in C_+(\overline{\Omega})$, we define

$$r_- := \min_{x\in\overline{\Omega}} r(x) \quad \text{and} \quad r_+ := \max_{x\in\overline{\Omega}} r(x).$$

Let $p \in C_+(\overline{\Omega})$. In what follows, we denote by $p' \in C_+(\overline{\Omega})$ the conjugate variable exponent to p, namely,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Furthermore, we denote by p^* the critical Sobolev variable exponent to p given by

$$p^*(x) := \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \quad \text{for all } x \in \overline{\Omega}. \tag{2.2}$$

For $r \in C_+(\overline{\Omega})$ the variable exponent Lebesgue space $L^{r(\cdot)}(\Omega)$ is defined by

$$L^{r(\cdot)}(\Omega) = \left\{ u \in M(\Omega) \,:\, \int_\Omega |u|^{r(x)} \,\mathrm{d}x < +\infty \right\}.$$

It is well-known that $L^{r(\cdot)}(\Omega)$ equipped with the Luxemburg norm given by

$$\|u\|_{r(\cdot)} := \inf \left\{ \lambda > 0 \,:\, \int_\Omega \left(\frac{|u|}{\lambda} \right)^{r(x)} \mathrm{d}x \leq 1 \right\},$$

is a separable and reflexive Banach space. Moreover, the dual space of $L^{r(\cdot)}(\Omega)$ is $L^{r'(\cdot)}(\Omega)$ and the following Hölder type inequality holds

$$\int_\Omega |uv| \,\mathrm{d}x \leq \left[\frac{1}{r_-} + \frac{1}{r'_-} \right] \|u\|_{r(\cdot)} \|v\|_{r'(\cdot)} \leq 2 \|u\|_{r(\cdot)} \|v\|_{r'(\cdot)}$$

for all $u \in L^{r(\cdot)}(\Omega)$ and for all $v \in L^{r'(\cdot)}(\Omega)$. If $r_1, r_2 \in C_+(\overline{\Omega})$ are such that $r_1(x) \leq r_2(x)$ for all $x \in \overline{\Omega}$, then we have the continuous embedding

$$L^{r_2(\cdot)}(\Omega) \hookrightarrow L^{r_1(\cdot)}(\Omega).$$

For any $r \in C_+(\overline{\Omega})$, we consider the modular function $\rho_{r(\cdot)} \colon L^{r(\cdot)}(\Omega) \to \mathbb{R}$ given by

$$\rho_{r(\cdot)}(u) = \int_\Omega |u|^{r(x)}\,\mathrm{d}x \quad \text{for all } u \in L^{r(\cdot)}(\Omega). \tag{2.3}$$

The following proposition states some important relations between the norm of $L^{r(\cdot)}(\Omega)$ and the modular function $\rho_{r(\cdot)}$ defined in (2.3).

Proposition 2.1. *If $r \in C_+(\overline{\Omega})$ and $u, u_n \in L^{r(\cdot)}(\Omega)$, then we have the following assertions:*

(i) $\|u\|_{r(\cdot)} = \lambda \iff \rho_{r(\cdot)}\left(\frac{u}{\lambda}\right) = 1$ *with* $u \neq 0$;
(ii) $\|u\|_{r(\cdot)} < 1$ (*resp.* $= 1$, > 1) $\iff \rho_{r(\cdot)}(u) < 1$ (*resp.* $= 1$, > 1);
(iii) $\|u\|_{r(\cdot)} < 1 \implies \|u\|_{r(\cdot)}^{r_+} \le \rho_{r(\cdot)}(u) \le \|u\|_{r(\cdot)}^{r_-}$;
(iv) $\|u\|_{r(\cdot)} > 1 \implies \|u\|_{r(\cdot)}^{r_-} \le \rho_{r(\cdot)}(u) \le \|u\|_{r(\cdot)}^{r_+}$;
(v) $\|u_n\|_{r(\cdot)} \to 0 \iff \rho_{r(\cdot)}(u_n) \to 0$;
(vi) $\|u_n\|_{r(\cdot)} \to +\infty \iff \rho_{r(\cdot)}(u_n) \to +\infty$.

For $r \in C_+(\overline{\Omega})$, we denote by $W^{1,r(\cdot)}(\Omega)$ the variable exponent Sobolev space given by

$$W^{1,r(\cdot)}(\Omega) = \left\{ u \in L^{r(\cdot)}(\Omega) \,:\, |\nabla u| \in L^{r(\cdot)}(\Omega) \right\}.$$

We know that $W^{1,r(\cdot)}(\Omega)$ equipped with the norm

$$\|u\|_{1,r(\cdot)} = \|u\|_{r(\cdot)} + \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W^{1,r(\cdot)}(\Omega)$$

is a separable and reflexive Banach space, where $\|\nabla u\|_{r(\cdot)} := \|\,|\nabla u|\,\|_{r(\cdot)}$. We also consider the subspace $W_0^{1,r(\cdot)}(\Omega)$ of $W^{1,r(\cdot)}(\Omega)$ defined by

$$W_0^{1,r(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,r(\cdot)}}.$$

For the space $W_0^{1,r(\cdot)}(\Omega)$, it is well-known that Poincaré's inequality holds, that is,

$$\|u\|_{r(\cdot)} \le c_0 \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega)$$

for some $c_0 > 0$. So, in what follows, we endow the space $W_0^{1,r(\cdot)}(\Omega)$ with the equivalent norm

$$\|u\|_{1,r(\cdot),0} = \|\nabla u\|_{r(\cdot)} \quad \text{for all } u \in W_0^{1,r(\cdot)}(\Omega).$$

For problem (1.1), in the whole paper, we assume that the weight function μ and the variable exponents p, q satisfy the following conditions:

(H1): $p, q \in C_+(\overline{\Omega})$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$ such that
(i) $p(x) < N$ for all $x \in \overline{\Omega}$;
(ii) $p(x) < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

We set $\mathbb{R}_+ := [0, +\infty)$. Let us introduce the nonlinear function $\mathcal{H} \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\mathcal{H}(x,t) = t^{p(x)} + \mu(x) t^{q(x)} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+.$$

In addition, we denote by $\rho_{\mathcal{H}}(\cdot)$ the modular function defined as

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, u(x))\, \mathrm{d}x = \int_\Omega \left(|u|^{p(x)} + \mu(x) |u|^{q(x)} \right) \mathrm{d}x. \tag{2.4}$$

In the sequel, $L^{\mathcal{H}}(\Omega)$ stands for the corresponding Musielak-Orlicz space related to the function $\mathcal{H}$ defined by

$$L^{\mathcal{H}}(\Omega) = \left\{ u \in M(\Omega) \,:\, \rho_{\mathcal{H}}(u) < +\infty \right\},$$

which is, equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} := \inf \left\{ \lambda > 0 \,:\, \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \leq 1 \right\} \quad \text{for all } u \in L^{\mathcal{H}}(\Omega),$$

a separable and reflexive Banach space. Similarly, we introduce the Musielak-Orlicz Sobolev spaces $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ given by

$$\begin{aligned} W^{1,\mathcal{H}}(\Omega) &= \left\{ u \in L^{\mathcal{H}}(\Omega) \,:\, |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\} \\ W^{1,\mathcal{H}}_0(\Omega) &= \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{1,\mathcal{H}}}, \end{aligned}$$

where the norm $\|\cdot\|_{1,\mathcal{H}}$ for both spaces is defined by

$$\|u\|_{1,\mathcal{H}} := \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W^{1,\mathcal{H}}(\Omega) \text{ resp. } W^{1,\mathcal{H}}_0(\Omega).$$

Furthermore, we introduce the seminormed space $L^{q(\cdot)}_\mu(\Omega)$ defined by

$$L^{q(\cdot)}_\mu(\Omega) := \left\{ u \in M(\Omega) \,:\, \int_\Omega \mu(x) |u|^{q(x)}\, \mathrm{d}x < +\infty \right\}$$

endowed with the seminorm

$$\|u\|_{q(\cdot),\mu} := \inf \left\{ \lambda > 0 \,:\, \int_\Omega \mu(x) \Big(\frac{|u|}{\lambda} \Big)^{q(x)} \mathrm{d}x \leq 1 \right\} \quad \text{for all } u \in L^{q(\cdot)}_\mu(\Omega).$$

From Crespo-Blanco, Gasiński, Harjulehto and Winkert [14, Proposition 2.13], we have the following proposition.

Proposition 2.2. *Let hypotheses* (H1) *be satisfied and let* $\rho_{\mathcal{H}}$ *be defined by* (2.4).

(i) *if* $u \neq 0$, *then* $\|u\|_{\mathcal{H}} = \lambda$ *if and only if* $\rho_{\mathcal{H}}(\frac{u}{\lambda}) = 1$;

(ii) $\|u\|_{\mathcal{H}} < 1$ (*resp.* $> 1, = 1$) *if and only if* $\rho_{\mathcal{H}}(u) < 1$ (*resp.* $> 1, = 1$);
(iii) *if* $\|u\|_{\mathcal{H}} < 1$, *then* $\|u\|_{\mathcal{H}}^{q_+} \leqslant \rho_{\mathcal{H}}(u) \leqslant \|u\|_{\mathcal{H}}^{p_-}$;
(iv) *if* $\|u\|_{\mathcal{H}} > 1$, *then* $\|u\|_{\mathcal{H}}^{p_-} \leqslant \rho_{\mathcal{H}}(u) \leqslant \|u\|_{\mathcal{H}}^{q_+}$;
(v) $\|u\|_{\mathcal{H}} \to 0$ *if and only if* $\rho_{\mathcal{H}}(u) \to 0$;
(vi) $\|u\|_{\mathcal{H}} \to +\infty$ *if and only if* $\rho_{\mathcal{H}}(u) \to +\infty$.

Next, we collect some useful embedding results for the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$. We refer to Crespo-Blanco, Gasiński, Harjulehto and Winkert [14, Proposition 2.15] for its detailed proof.

Proposition 2.3. *Let hypotheses* (H1) *be satisfied and let* $p^*(\cdot)$ *be the critical exponent to* $p(\cdot)$ *given in* (2.2). *Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, $W^{1,\mathcal{H}}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega)$, $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r(\cdot)}(\Omega)$ *are continuous for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) \leq p(x)$ *for all* $x \in \Omega$;
(ii) $W^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *and* $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ *are compact for all* $r \in C(\overline{\Omega})$ *with* $1 \leq r(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$;
(iii) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^{q(\cdot)}_{\mu}(\Omega)$ *is continuous;*
(iv) $L^{q(\cdot)}(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

From Proposition 2.16 (ii) in Crespo-Blanco, Gasiński, Harjulehto and Winkert [14], we know that Poincaré's inequality holds

$$\|u\|_{\mathcal{H}} \leq c_1 \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega)$$

for some $c_1 > 0$ independent of u. Therefore, in this paper, we equip $W_0^{1,\mathcal{H}}(\Omega)$ with the equivalent norm

$$\|u\| = \|\nabla u\|_{\mathcal{H}} \quad \text{for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

Throughout the paper the symbols " $\xrightarrow{w}$ " and "$\to$" stand for the weak and the strong convergences, respectively. For a Banach space $(X, \|\cdot\|_X)$, we denote its dual space by X^* and by $\langle \cdot,\cdot\rangle_{X^*\times X}$ the duality pairing between X^* and X.

Now, we consider the nonlinear operator $A\colon W_0^{1,\mathcal{H}}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ defined by

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x \tag{2.5}$$

for $u, v \in W_0^{1,\mathcal{H}}(\Omega)$ with $\langle\cdot,\cdot\rangle$ being the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$. The operator has the following properties, see Crespo-Blanco, Gasiński, Harjulehto and Winkert [14, Theorem 3.3].

Theorem 2.4. *Let hypotheses* (H1) *be satisfied. Then, the operator A defined by* (2.5) *is bounded, continuous, strictly monotone and of type* (S_+)*, that is,*

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0,$$

imply $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

Next, we recall the following definition about Kuratowski limits, see Papageorgiou and Winkert [40, Definition 6.7.4].

Definition 2.5. *Let (X,τ) be a Hausdorff topological space and let $\{A_n\} \subset 2^X$ be a sequence of sets. We define the τ-Kuratowski lower limit of the sets A_n by*

$$\tau\text{-}\liminf_{n\to\infty} A_n := \left\{ x \in X \,:\, x = \tau\text{-}\lim_{n\to\infty} x_n,\, x_n \in A_n \text{ for all } \ n \geq 1 \right\},$$

and the τ-Kuratowski upper limit of the sets A_n

$$\tau\text{-}\limsup_{n\to\infty} A_n := \Big\{ x \in X : x = \tau\text{-}\lim_{k\to\infty} x_{n_k}, x_{n_k} \in A_{n_k}, \\ n_1 < n_2 < \ldots < n_k < \ldots \Big\}.$$

If

$$A = \tau\text{-}\liminf_{n\to\infty} A_n = \tau\text{-}\limsup_{n\to\infty} A_n,$$

then A is called τ-Kuratowski limit of the sets A_n.

We end this section by recalling the following existence theorem to mixed variational inequalities which will be applied in Section 3 for examining the nonemptiness of the solution set to problem (1.1), see, for example, Theorem 3.1 of Khan and Motreanu [29].

Theorem 2.6. *Let X be a reflexive Banach space with its dual space X^*, C be a nonempty, bounded, closed and convex subset of X, $f \in X^*$. If $\Psi\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is l.s.c. and convex with $C \cap D(\Psi) \neq \emptyset$, and multivalued mapping $F\colon X \to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $x \in X$, the set $F(x)$ is nonempty, closed and convex in X^*,*
(ii) *for any sequence $\{(x_n, w_n)\}_{n\in\mathbb{N}} \subset \mathrm{Gr}(F)$ such that*

$$x_n \xrightarrow{w} x \quad \text{and} \quad \limsup_{n\to\infty}\langle w_n, x_n - x\rangle \leq 0,$$

then for each $y \in X$ *there exists* $w(y) \in F(x)$ *satisfying*

$$\liminf_{n\to\infty} \langle w_n, x_n - y\rangle \geq \langle w(y), x - y\rangle,$$

(ii) *for each* $x \in X$ *and for each bounded subset* B *of* X *with* $B \cap D(F) \neq \emptyset$, *there exists a constant* $c(B, x) \in \mathbb{R}$ *such that for each* $(z, u) \in \mathrm{Gr}(F)$ *with* $z \in B$ *it holds*

$$\langle u, z - x\rangle \geq c(B, x),$$

then there exists $x \in C \cap D(\Psi)$ *such that for some* $w \in F(x)$, *we have*

$$\langle w - f, z - x\rangle + \Psi(z) - \Psi(x) \geq 0 \quad \textit{for all } z \in C.$$

3. Properties of the solution set

This section is devoted to explore the properties of the solution set to problem (1.1) which turns out to be nonempty, bounded and closed. First we impose the following hypotheses on the data of problem (1.1).

(H2): $\theta \in C_+(\overline{\Omega})$ is such that

$$p_- < \theta(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega},$$

where p^* is the critical Sobolev variable exponent of p given in (2.2).

(H3): The multivalued convection mapping $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ has nonempty, compact and convex values such that $f(x, 0, 0) \neq \{0\}$ for a. a. $x \in \Omega$ and

(i) the multivalued mapping $x \mapsto f(x, s, \xi)$ has a measurable selection for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$;

(ii) the multivalued mapping $(s, \xi) \mapsto f(x, s, \xi)$ is upper semicontinuous for a. a. $x \in \Omega$;

(iii) there exist $\alpha_f \in L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega)_+$ and $a_f, b_f \geq 0$ such that

$$|\eta| \leq a_f |\xi|^{\frac{p(x)(r(x)-1)}{r(x)}} + b_f |s|^{r(x)-1} + \alpha_f(x)$$

for all $\eta \in f(x, s, \xi)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$ and for a. a. $x \in \Omega$, where $r \in C_+(\overline{\Omega})$ is such that

$$r(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega};$$

(iv) there exist $\beta_f \in L^1_+(\Omega)$ and $c_f, d_f \geq 0$ satisfying

$$\eta s \leq c_f |\xi|^{p(x)} + d_f |s|^{p_-} + \beta_f(x)$$

for all $\eta \in f(x,s,\xi)$, for all $s \in \mathbb{R}$, for all $\xi \in \mathbb{R}^N$ and for a. a. $x \in \Omega$ and the inequality

$$1 - c_f - d_f \lambda_{1,p_-}^{-1} > 0,$$

holds, where λ_{1,p_-} is the first eigenvalue of the Dirichlet eigenvalue problem for the p_--Laplacian, see (2.1) for $s = p_-$.

(H4): The function $\Phi\colon \Omega \to [0,\infty)$ is such that $\Phi \in M(\Omega)$.

Let K be defined by

$$K = \left\{ u \in W_0^{1,\mathcal{H}}(\Omega) \,:\, u(x) \leq \Phi(x) \text{ for a. a. } x \in \Omega \right\}. \tag{3.1}$$

Remark 3.1. Let (H4) be satisfied. Then, the set K is a nonempty, closed and convex subset of $W_0^{1,\mathcal{H}}(\Omega)$. Indeed, since $\Phi(x) \geq 0$ for a. a. $x \in \Omega$, we see that $0 \in K$, i.e., $K \neq \emptyset$. The convexity of K is obvious. Let $\{u_n\}_{n\in\mathbb{N}} \subset K$ be a sequence such that $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$ for some $u \in W_0^{1,\mathcal{H}}(\Omega)$. Since the embedding of $W_0^{1,\mathcal{H}}(\Omega)$ to $L^{p_-}(\Omega)$ is continuous, we have $u_n \to u$ in $L^{p_-}(\Omega)$ as $n \to \infty$. Passing to a subsequence if necessary, we may assume that $u_n(x) \to u(x)$ as $n \to \infty$ for a. a. $x \in \Omega$. Hence, it follows that

$$\Phi(x) \geq \lim_{n\to\infty} u_n(x) = u(x) \quad \text{for a. a. } x \in \Omega,$$

thus, $u \in K$. Therefore, K is closed.

We are now in a position to give the following definition of weak solutions to problem (1.1).

Definition 3.2. *A function $u \in K$ is called a weak solution of problem* (1.1) *if there exists a function $\eta \in L^{r'(\cdot)}(\Omega)$ such that*

$$\eta(x) \in f(x, u(x), \nabla u(x))$$

for a. a. $x \in \Omega$ and

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u \right) \cdot \nabla(v-u)\,\mathrm{d}x \\ &\qquad + \int_\Omega \beta |u|^{\theta(x)-2} u (v-u)\,\mathrm{d}x \\ &\geq \int_\Omega \eta(x)(v-u)\,\mathrm{d}x \end{aligned}$$

for all $v \in K$ with K defined in (3.1).

The main result in this section is given by the following theorem which states several properties of the solution set of problem (1.1).

Theorem 3.3. *Let hypotheses* (H1)–(H4) *be satisfied. Then, the solution set* $\mathcal{S}$ *of problem* (1.1) *is nonempty, bounded and weakly closed in* $W_0^{1,\mathcal{H}}(\Omega)$ (*hence, weakly compact in* $W_0^{1,\mathcal{H}}(\Omega)$).

Proof. Existence: From hypotheses (H3)(i), (ii) and the Yankov-von Neumann-Aumann selection theorem (see Papageorgiou and Winkert [40, Theorem 2.7.25]) it follows that for each $u \in W_0^{1,\mathcal{H}}(\Omega)$, we are able to find a measurable selection $\eta\colon \Omega \to \mathbb{R}$ such that $\eta(x) \in f(x, u(x), \nabla u(x))$ for a. a. $x \in \Omega$. On the other hand, employing hypothesis (H3)(iii) and the elementary inequality

$$(|a| + |b|)^s \leq 2^{s-1}(|a|^s + |b|^s)$$

for all $s \geq 1$ and for all $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned} &\int_\Omega |\eta(x)|^{r(x)'} \,\mathrm{d}x \leq \int_\Omega \left(a_f |\nabla u|^{\frac{p(x)}{r(x)'}} + b_f |u|^{r(x)-1} + \alpha_f(x)\right)^{r(x)'} \mathrm{d}x \qquad (3.2)\\ &\leq M_1 \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{r(x)} + \alpha_f(x)^{r(x)'}\right) \mathrm{d}x \\ &= M_1 \Big(\rho_{p(\cdot)}(|\nabla u|) + \rho_{r(\cdot)}(u) + \rho_{r'(\cdot)}(\alpha_f)\Big) \\ &\leq M_1 \Big(\max\left\{\|\nabla u\|_{p(\cdot)}^{p_-}, \|\nabla u\|_{p(\cdot)}^{p_+}\right\} + \max\left\{\|u\|_{r(\cdot)}^{r_-}, \|u\|_{r(\cdot)}^{r_+}\right\} \\ &\qquad + \max\left\{\|\alpha_f\|_{r'(\cdot)}^{r'_-}, \|\alpha_f\|_{r'(\cdot)}^{r'_+}\right\}\Big) \\ &< +\infty, \end{aligned}$$

for some $M_1 > 0$, where we have used Proposition 2.1(iii), (iv) and the inequality

$$\int_\Omega c_2^{r'(x)} \,\mathrm{d}x \leq \max\left\{|\Omega| c_2^{r_-}, |\Omega| c_2^{r_+}\right\} \quad \text{for any } c_2 > 0.$$

So, we have $\eta \in L^{r'(\cdot)}(\Omega)$. This allows us to introduce the Nemytskij operator $\mathcal{N}_f\colon W_0^{1,\mathcal{H}}(\Omega) \subset L^{r(\cdot)}(\Omega) \to 2^{L^{r(\cdot)'}(\Omega)}$ corresponding to the multivalued mapping f given by

$$\mathcal{N}_f(u) = \left\{\eta \in L^{r(\cdot)'}(\Omega) \,:\, \eta(x) \in f(x, u(x), \nabla u(x)) \text{ for a. a. } x \in \Omega\right\}$$

for all $u \in W_0^{1,\mathcal{H}}(\Omega)$.

Let $\iota\colon W_0^{1,\mathcal{H}}(\Omega) \to X := L^{r(\cdot)}(\Omega)$ be the embedding operator of $W_0^{1,\mathcal{H}}(\Omega)$ into X. It is obvious from Proposition 2.3(ii) that ι is linear and compact.

We denote by $\iota^*\colon L^{r'(\cdot)}(\Omega) \to W_0^{1,\mathcal{H}}(\Omega)^*$ the adjoint operator of ι and consider the nonlinear operator $B\colon L^{\theta(\cdot)}(\Omega) \to L^{\theta'(\cdot)}(\Omega)$ defined by

$$\langle Bu, v\rangle = \int_\Omega \beta |u|^{\theta(x)-2} uv \,\mathrm{d}x \quad \text{for all } u, v \in L^{\theta(\cdot)}(\Omega).$$

Obviously, B is a bounded, continuous and monotone operator. Under the definitions above, it is not difficult to see that $u \in K$ is a solution of problem (1.1), if there exists $\eta \in L^{r'(\cdot)}(\Omega)$ such that $\eta(x) \in f(x, u(x), \nabla u(x))$ for a. a. $x \in \Omega$ and

$$\langle Au + Bu, v - u\rangle \geq \langle \eta, v - u\rangle$$

for all $v \in K$.

Let $n \in \mathbb{N}$ be large enough such that $K_n := K \cap \overline{B(0,n)} \neq \emptyset$, where $\overline{B(0,n)}$ is the closed ball centered at the origin with radius n. First, we consider the following auxiliary problem: Find $u_n \in K_n$ such that there exists $\eta_n \in \mathcal{N}_f(u_n)$ and

$$\langle Au_n + Bu_n, v - u_n\rangle \geq \langle \eta_n, v - u_n\rangle \tag{3.3}$$

for all $v \in K_n$. We are going to apply Theorem 2.6 to examine the existence of a solution to problem (3.3). To this end, let

$$Fu := Au + Bu - \iota^* \mathcal{N}_f(u) \ \text{ for all } u \in W_0^{1,\mathcal{H}}(\Omega).$$

From Kenmochi [28], we see that if F is pseudomonotone, then F satisfies all conditions of Theorem 2.6.

We claim that $F\colon W_0^{1,\mathcal{H}}(\Omega) \to 2^{W_0^{1,\mathcal{H}}(\Omega)^*}$ is pseudomonotone and coercive. The boundedness of A and B along with (3.2) implies that F is a bounded mapping. The convexity of f guarantees that $\mathcal{N}_f(u)$ is also convex for each $u \in W_0^{1,\mathcal{H}}(\Omega)$. This means that $F(u)$ is nonempty, bounded, closed and convex in $W_0^{1,\mathcal{H}}(\Omega)^*$ for each $u \in W_0^{1,\mathcal{H}}(\Omega)$. Taking Proposition 3.58 of Migórski, Ochal and Sofonea [34] into account, it is sufficient to prove that F is a generalized pseudomonotone operator. Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ and $\{\eta_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)^*$ be sequences such that

$$\begin{aligned} &u_n \xrightarrow{w} u \quad \text{in } W_0^{1,\mathcal{H}}(\Omega), \quad \eta_n \xrightarrow{w} \eta \quad \text{in } W_0^{1,\mathcal{H}}(\Omega)^*,\\ &\eta_n \in F(u_n) \quad \text{and} \quad \limsup_{n\to\infty} \langle \eta_n, u_n - u\rangle \leq 0 \end{aligned} \tag{3.4}$$

for some $u \in W_0^{1,\mathcal{H}}(\Omega)$. Our goal is to show that $\eta \in F(u)$ and $\langle \eta_n, u_n\rangle \to \langle \eta, u\rangle$. For each $n \in \mathbb{N}$, there exists $\xi_n \in \mathcal{N}_f(u_n)$ such that

$$\eta_n = Au_n + Bu_n - \iota^* \xi_n$$

for each $n \in \mathbb{N}$. Keeping in mind that $\mathcal{N}_f$ is a bounded mapping (see (3.2)), we conclude that $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded in $L^{r'(\cdot)}(\Omega)$. Passing to a subsequence if necessary, we may assume that $\xi_n \xrightarrow{w} \xi$ in $L^{r'(\cdot)}(\Omega)$ as $n \to \infty$ for some $\xi \in L^{r'(\cdot)}(\Omega)$. From (3.4), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle + \liminf_{n\to\infty}\langle Bu_n, u_n - u\rangle + \liminf_{n\to\infty}\langle \iota^*\xi_n, u - u_n\rangle \\
&\le \limsup_{n\to\infty}\langle \eta_n, u_n - u\rangle \le 0.
\end{aligned} \tag{3.5}$$

Note that ι is a compact embedding, so it holds

$$\lim_{n\to\infty}\langle \iota^*\xi_n, u - u_n\rangle = \lim_{n\to\infty}\langle \xi_n, \iota(u - u_n)\rangle_{L^{r'(\cdot)}(\Omega)\times L^{r(\cdot)}(\Omega)} = 0. \tag{3.6}$$

Because of $\theta(x) < p^*(x)$ for all $x \in \overline{\Omega}$, we use Proposition 2.3 (ii) in order to find

$$\lim_{n\to\infty}\langle Bu_n, u_n - u\rangle = \lim_{n\to\infty}\langle Bu_n, u_n - u\rangle_{L^{\theta'(\cdot)}(\Omega)\times L^{\theta(\cdot)}(\Omega)} = 0. \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5) yields

$$\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0.$$

This combined with the fact that A is of type (S_+) (see Theorem 2.4) implies $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Taking into account the continuity of A and B, we know that

$$Au_n \to Au \quad \text{and} \quad Bu_n \to Bu.$$

Employing Mazur's theorem for the sequence $\{\xi_n\}_{n\in\mathbb{N}}$, we know that there exists a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of convex combinations of $\{\xi_n\}_{n\in\mathbb{N}}$ such that

$$\zeta_n \to \xi \quad \text{in } L^{r'(\cdot)}(\Omega).$$

Since the embeddings of $L^{r'(\cdot)}(\Omega)$ into $L^{r'_-}(\Omega)$ and of $W_0^{1,\mathcal{H}}(\Omega)$ into $W^{1,r_-}(\Omega)$ are both continuous, we may assume, without any loss of generality, that

$$\begin{gathered}
\zeta_n(x) \to \xi(x), \quad u_n(x) \to u(x) \\
\text{and} \\
\nabla u_n(x) \to \nabla u(x) \quad \text{for a.a. } x \in \Omega.
\end{gathered} \tag{3.8}$$

However, the convexity of f ensures that

$$\zeta_n(x) \in f(x, u_n(x), \nabla u_n(x))$$

for a. a. $x \in \Omega$. Since f is u.s.c. and has nonempty, bounded, closed values, using Theorem 1.1.4 of Kamenskii, Obukhovskii and Zecca [27], we conclude that f is closed, that is, f has closed graph. This fact along with the convergence properties in (3.8) shows that $\xi(x) \in f(x, u(x), \nabla u(x))$ for a. a. $x \in \Omega$. This reveals that $\xi \in \mathcal{N}_f(u)$. Therefore, we have

$$\eta = Au + Bu - \iota^* \xi \in F(u)$$

and

$$\lim_{n\to\infty} \langle \eta_n, u_n \rangle = \lim_{n\to\infty} \langle Au_n + Bu_n - \iota\xi_n, u_n \rangle = \langle Au + Bu - \iota\xi, u \rangle = \langle \eta, u \rangle.$$

Consequently, we have proved that F is generalized pseudomonotone. Applying Proposition 3.58 of Migórski, Ochal and Sofonea [34], it turns out that F is pseudomonotone.

Next, we have to show that F is coercive. For any $u \in W_0^{1,\mathcal{H}}(\Omega)$ and any $\eta \in \mathcal{N}_f(u)$, by using Proposition 2.2(iii) and (iv), we have

$$\begin{aligned}
\langle Fu, u \rangle &= \int_\Omega \left(|\nabla u|^{p(x)} + \mu(x) |\nabla u|^{q(x)} \right) \mathrm{d}x + \int_\Omega \beta |u|^{\theta(x)} \,\mathrm{d}x - \int_\Omega \eta(x) u \,\mathrm{d}x \\
&\geq \rho_{\mathcal{H}}(\nabla u) - \int_\Omega \left(c_f |\nabla u|^{p(x)} + d_f |u|^{p_-} + \beta_f(x) \right) \mathrm{d}x \\
&\geq \rho_{\mathcal{H}}(\nabla u) - c_f \rho_{\mathcal{H}}(\nabla u) - d_f \lambda_{1,p_-}^{-1} \|\nabla u\|_{p_-}^{p_-} + \|\beta_f\|_1 \\
&\geq \rho_{\mathcal{H}}(\nabla u) - c_f \rho_{\mathcal{H}}(\nabla u) - d_f \lambda_{1,p_-}^{-1} \int_\Omega |u|^{p(x)} \,\mathrm{d}x - \|\beta_f\|_1 - M_3 \\
&\geq \left(1 - c_f - d_f \lambda_{1,p_-}^{-1} \right) \rho_{\mathcal{H}}(\nabla u) - \|\beta_f\|_1 - M_3 \qquad (3.9) \\
&\geq \left(1 - c_f - d_f \lambda_{1,p_-}^{-1} \right) \min\left\{ \|\nabla u\|_{\mathcal{H}}^{q_+}, \|\nabla u\|_{\mathcal{H}}^{p_-} \right\} - \|\beta_f\|_1 - M_3.
\end{aligned}$$

Here, we have used Young's inequality to get

$$\int_\Omega |u|^{p_-} \,\mathrm{d}x \leq \int_\Omega |u|^{p(x)} \,\mathrm{d}x + M_3$$

for some $M_3 > 0$ owing to $p \in C(\overline{\Omega})$ with $p(x) \geq p_-$ for all $x \in \Omega$. This proves that F is coercive.

Therefore, all conditions of Theorem 2.6 are fulfilled with $\Psi \equiv 0$. Using this theorem, we conclude that for each $n \in \mathbb{N}$ problem (3.3) has at least one solution $u_n \in K_n$. Furthermore, we claim that there exists $N_0 > 0$ such that

$$\|u_{N_0}\| < N_0, \qquad (3.10)$$

where u_{N_0} is a solution of problem (3.3) with $n = N_0$. Let us assume that (3.10) is not true. Then for each $n \in \mathbb{N}$ and for any solution $u_n \in K_n$ of problem (3.3), we have

$$\|u_n\| = n. \tag{3.11}$$

Since $0 \in K_n$ for every $n \in \mathbb{N}$, we can take $v = 0$ into (3.3) in order to get

$$\langle Au_n + Bu_n, u_n \rangle \leq \langle \eta_n, u_n \rangle.$$

From (3.9), we have

$$\begin{aligned}
\rho_{\mathcal{H}}(\nabla u_n) &\leq \int_\Omega \left(c_f |\nabla u_n|^{p(x)} + d_f |u_n|^{p_-} + \beta_f(x) \right) \mathrm{d}x \\
&\leq c_f \rho_{\mathcal{H}}(\nabla u_n) + d_f \lambda_{1,p_-}^{-1} \|\nabla u_n\|_{p_-}^{p_-} + \|\beta_f\|_1 \\
&\leq c_f \rho_{\mathcal{H}}(\nabla u_n) + d_f \lambda_{1,p_-}^{-1} \int_\Omega |u_n|^{p(x)} \, \mathrm{d}x + \|\beta_f\|_1 + M_4 \\
&\leq \left(c_f + d_f \lambda_{1,p_-}^{-1} \right) \rho_{\mathcal{H}}(\nabla u_n) + \|\beta_f\|_1 + M_4
\end{aligned} \tag{3.12}$$

for some $M_4 > 0$. Hence,

$$\begin{aligned}
&\left(1 - c_f - d_f \lambda_{1,p_-}^{-1}\right) \min\left\{ \|\nabla u_n\|_{\mathcal{H}}^{q_+}, \|\nabla u_n\|_{\mathcal{H}}^{p_-} \right\} \\
&\leq \left(1 - c_f - d_f \lambda_{1,p_-}^{-1}\right) \rho_{\mathcal{H}}(\nabla u_n) \\
&\leq \|\beta_f\|_1 + M_4.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the inequality above and using (3.11) this leads to a contradiction. Therefore, there exists $N_0 > 0$ such that inequality (3.10) holds. Let $u_{N_0} \in K_{N_0}$ satisfy inequality (3.10). For any $w \in K$, we take $t \in (0,1)$ small enough such that

$$v_t = tw + (1-t)u_{N_0} \in K_{N_0}$$

which is possible due to (3.10). Inserting v_t into (3.3) with $n = N_0$ gives

$$\langle Au_{N_0} + Bu_{N_0}, w - u_{N_0} \rangle \geq \langle \eta_{N_0}, w - u_{N_0} \rangle$$

with $\eta_{N_0} \in \mathcal{N}_f(u_{N_0})$.

The arbitrariness of $w \in K$ and the fact that $f(x,0,0) \neq \{0\}$ for a. a. $x \in \Omega$, implies that u_{N_0} is a nontrivial weak solution of problem (1.1). Consequently, the solution set of problem (1.1) is nonempty.

Boundedness: Arguing by contradiction, suppose that the solution set $\mathcal{S}$ of problem (1.1) is unbounded. Then, we are able to find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\|u_n\| \to \infty$ as $n \to \infty$. Arguing as above, for each $n \in \mathbb{N}$, we have $\eta_n \in L^{r'(\cdot)}(\Omega)$ with $\eta_n(x) \in f(x, u_n(x), \nabla u_n(x))$ for a. a. $x \in \Omega$ and

$$\int_\Omega \left(|\nabla u_n|^{p(x)} + \mu(x)|\nabla u_n|^{q(x)}\right) \,\mathrm{d}x + \int_\Omega \beta |u_n|^{\theta(x)} \,\mathrm{d}x - \int_\Omega \eta_n(x) u_n \,\mathrm{d}x \leq 0.$$

Applying (3.9) yields

$$0 \geq \left(1 - c_f - d_f \lambda_{1,p_-}^{-1}\right) \min\{\|\nabla u_n\|_{\mathcal{H}}^{q_+}, \|\nabla u_n\|_{\mathcal{H}}^{p_-}\} - \|\beta_f\|_1 - M_5 \to +\infty$$

for some $M_5 > 0$, which is a contradiction. Therefore, the solution set $\mathcal{S}$ of problem (1.1) is bounded.

Closedness: Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ be a sequence such that $u_n \xrightarrow{w} u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Then, for each $n \in \mathbb{N}$, there exists $\eta_n \in L^{r'(\cdot)}(\Omega)$ such that $\eta_n(x) \in f(x, u_n(x), \nabla u_n(x))$ for a. a. $x \in \Omega$ and

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right) \cdot \nabla(v - u_n) \,\mathrm{d}x \\ &+ \int_\Omega \beta |u_n|^{\theta(x)-2} u_n (v - u_n) \,\mathrm{d}x \geq \int_\Omega \eta_n(x)(v - u_n) \,\mathrm{d}x \end{aligned} \tag{3.13}$$

for all $v \in K$. The convexity and the closedness of K ensures that $u \in K$. Recall that the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ is compact and the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in $L^{r'(\cdot)}(\Omega)$ (see (3.2)). Therefore, we have

$$\lim_{n\to\infty} \int_\Omega \eta_n(x)(u - u_n) \,\mathrm{d}x = 0,$$

where we have used Fatou's Lemma. Taking $v = u$ in (3.13) and passing to the upper limit as $n \to \infty$ for the resulting inequality, we get

$$\begin{aligned} &\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \\ &\leq \limsup_{n\to\infty} \langle B(u_n), u - u_n\rangle - \lim_{n\to\infty} \int_\Omega \eta_n(x)(u - u_n) \,\mathrm{d}x \leq 0. \end{aligned}$$

Here we have applied the continuity of B and the compactness of the embedding of $W_0^{1,\mathcal{H}}(\Omega)$ into $L^{\theta(\cdot)}(\Omega)$. This together with the convergence $u_n \xrightarrow{w} u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and the (S_+)-property of A (see Proposition 2.4) deduces that $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.

From hypotheses (H3) and the boundedness of $\{\eta_n\}_{n\in\mathbb{N}}$, we can show that $\eta_n \xrightarrow{w} \eta$ in $L^{r'(\cdot)}(\Omega)$ with some $\eta \in L^{r'(\cdot)}(\Omega)$ such that $\eta(x) \in$

$f(x,u(x),\nabla u(x))$ for a. a. $x\in\Omega$. Taking the upper limit in inequality (3.13) as $n\to\infty$ yields

$$\begin{aligned}&\int_\Omega\left(|\nabla u|^{p(x)-2}\nabla u+\mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot\nabla(v-u)\,\mathrm{d}x\\&\qquad+\int_\Omega\beta|u|^{\theta(x)-2}u(v-u)\,\mathrm{d}x\\&\geq\int_\Omega\eta(x)(v-u)\,\mathrm{d}x\end{aligned}$$

for all $v\in K$ with $\eta(x)\in f(x,u(x),\nabla u(x))$ for a. a. $x\in\Omega$. Thus, $u\in\mathcal{S}$ and so, $\mathcal{S}$ is weakly closed in $W_0^{1,\mathcal{H}}(\Omega)$. $\square$

Let us now mention some special cases of our problem. Particularly, if $\Phi\equiv+\infty$, then we have $K=W_0^{1,\mathcal{H}}(\Omega)$. In this situation, we can use the same arguments as in the proof of Theorem 3.3 to get the following result.

Corollary 3.4. *Let hypotheses* (H1)–(H3) *be satisfied. Then, the solution set of the elliptic inclusion* (1.3) *is nonempty, bounded and weakly closed in* $W_0^{1,\mathcal{H}}(\Omega)$.

If $\beta=0$, we have the following existence theorem for problem (1.2).

Corollary 3.5. *Let hypotheses* (H1), (H3) *and* (H4) *be satisfied. Then, the solution set of the elliptic inclusion* (1.2) *is nonempty, bounded and weakly closed in* $W_0^{1,\mathcal{H}}(\Omega)$.

If we combine the two cases above, that is, $\beta=0$ and $\Phi\equiv+\infty$, we obtain the following result.

Corollary 3.6. *Let hypotheses* (H1) *and* (H3) *be satisfied. Then, the solution set of the elliptic inclusion* (1.4) *is nonempty, bounded and weakly closed in* $W_0^{1,\mathcal{H}}(\Omega)$.

Remark 3.7. Note that if f is a single valued mapping, then Corollary 3.6 coincides with Theorem 4.4 of Crespo-Blanco, Gasiński, Harjulehto and Winkert [14].

Finally, when p,q and θ are constants, that is, $1<p<N$, $p<q<p^*$ and $1<\theta<p^*$, then we have the following result.

Corollary 3.8. *Let hypotheses* (H3) *and* (H4) *be satisfied. If, in addition,* p,q,θ *are constants and* $0\leq\mu(\cdot)\in L^\infty(\Omega)$ *such that* $1<p<N$, $p<q<p^*$ *and* $\theta<p^*$, *then the solution set of the elliptic inclusion* (1.5) *is nonempty, bounded and weakly closed in* $W_0^{1,\mathcal{H}}(\Omega)$.

4. Convergence analysis

This section is devoted to explore a critical convergence result for the variable exponent double phase obstacle problem given in (1.1). More precisely, when the obstacle function Φ is approximated by a suitable sequence, via applying a generalized penalty technique, we are going to introduce a family of perturbed problems without constraints associated with problem (1.1). Then, a convergence theorem is established which shows that the solution set $\mathcal{S}$ can be approached by the solution sets of the perturbed problems, denoted by $\{\mathcal{S}_n\}_{n\in\mathbb{N}}$, in the sense of Kuratowski.

We suppose the following assumptions.

(H5): $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a sequence such that $\varepsilon_n > 0$ for each $n \in \mathbb{N}$ and $\varepsilon_n \to 0$ as $n \to \infty$.

(H6): $\Phi \in W_0^{1,\mathcal{H}}(\Omega)$ and $\{\Phi_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ are such that $\Phi(x) > 0$ for a. a. $x \in \Omega$ and $\Phi_n \to \Phi$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$.

From hypothesis (H6), without any loss of any generality, we can assume that $\Phi_n(x) \geq 0$ for a. a. $x \in \Omega$ and for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let us introduce a family of penalty operators $\{P_n\}_{n\in\mathbb{N}}$ with $P_n \colon L^{p(\cdot)}(\Omega) \to L^{p'(\cdot)}(\Omega)$ associated to the sets $\{K_n\}_{n\in\mathbb{N}}$ defined by

$$\langle P_n u, v\rangle_{L^{p'(\cdot)}(\Omega)\times L^{p(\cdot)}(\Omega)} = \int_\Omega \left[(u-\Phi_n)^+\right]^{p(x)-1} v \,\mathrm{d}x \tag{4.1}$$

for all $u, v \in L^{p(\cdot)}(\Omega)$, where K_n is given by

$$K_n = \left\{u \in W_0^{1,\mathcal{H}}(\Omega) \,:\, u(x) \leq \Phi_n(x) \text{ for a. a. } x \in \Omega\right\}.$$

For each fixed $n \in \mathbb{N}$, the following lemma gives some important properties of P_n.

Lemma 4.1. *If $\Phi_n \in L^{p'(\cdot)}(\Omega)$, then the function*

$$P_n \colon L^{p(\cdot)}(\Omega) \to L^{p'(\cdot)}(\Omega)$$

given in (4.1) *is bounded, demicontinuous and monotone.*

Proof. Let $u \in L^{p(\cdot)}(\Omega)$ and $\lambda = \|u\|_{p(\cdot)}$. From Young's inequality and Proposition 2.1 it follows that

$$\begin{aligned}\|P_n u\|_{p'(\cdot)} &= \sup_{v\in L^{p(\cdot)}(\Omega),\, \|v\|_{p(\cdot)}=1} \langle P_n u, v\rangle_{L^{p'(\cdot)}(\Omega)\times L^{p(\cdot)}(\Omega)} \\ &= \sup_{v\in L^{p(\cdot)}(\Omega),\, \|v\|_{p(\cdot)}=1} \int_\Omega \Big[\,(u-\Phi_n)^+\,\Big]^{p(x)-1} v\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{v\in L^{p(\cdot)}(\Omega),\,\|v\|_{p(\cdot)}=1} \left[\frac{p_+-1}{p_-}\int_\Omega \left[(u-\Phi_n)^+\right]^{p(x)}\,\mathrm{d}x+\frac{1}{p_-}\int_\Omega |v|^{p(x)}\,\mathrm{d}x\right]\\
&=\left[\frac{p_+-1}{p_-}\int_\Omega \left[(u-\Phi_n)^+\right]^{p(x)}\,\mathrm{d}x+\frac{1}{p_-}\right]\\
&\leq \left[\frac{p_+-1}{p_-}\int_\Omega \left[|u|+|\Phi_n|\right]^{p(x)}\,\mathrm{d}x+\frac{1}{p_-}\right]\\
&\leq \left[\frac{(p_+-1)M_6}{p_-}\int_\Omega \left[\left(\frac{|u|}{\lambda}\right)^{p(x)}\lambda^{p(x)}+|\Phi_n|^{p(x)}\right]\mathrm{d}x+\frac{1}{p_-}\right]\\
&\leq M_7\left[\max\left\{\lambda^{p_-},\lambda^{p_+}\right\}+1\right]
\end{aligned}
$$

for some $M_6, M_7>0$. Therefore, P_n is bounded.

Note that $D(P_n)=L^{p(\cdot)}(\Omega)$. So, from Denkowski, Migórski and Papageorgiou [16, Exercise I.9 in Sect. 1.9], we know that P_n is demicontinuous if and only if P_n is hemicontinuous. Employing the estimates above along with Lebesgue's Dominated Convergence Theorem, it is not difficult to see that $t\mapsto \langle P_n(u+tv),w\rangle$ is continuous for all $u,v,w\in L^{p(\cdot)}(\Omega)$. Thus, P_n is hemicontinuous and so it is also demicontinuous. Finally, the monotonicity of P_n is a direct consequence of the fact that the function $s\mapsto (s^+)^\eta$ is increasing for all $\eta>0$. $\square$

Remark 4.2. From the definition of K_n and P_n, we have $P_n u=0$ for all $u\in K_n$, that is, $K_n\subset \ker(P_n)$. It is not difficult to see that if $u\in L^{p(\cdot)}(\Omega)$ is such that $P_n u=0$, then we have

$$
\left[(u-\Phi_n)^+\right]^{p(x)-1}=0\quad \text{for a.\,a. } x\in\Omega.
$$

This implies that $u(x)\leq \Phi_n(x)$ a. a. $x\in\Omega$. Therefore, $u\in W_0^{1,\mathcal{H}}(\Omega)$ with $P_n u=0$ entails that $u\in K_n$, i.e., $\ker(P_n)=K_n$.

We introduce the function $P\colon L^{p(\cdot)}(\Omega)\to L^{p'(\cdot)}(\Omega)$ given by

$$
\langle Pu,v\rangle_{L^{p'(\cdot)}(\Omega)\times L^{p(\cdot)}(\Omega)}=\int_\Omega \left[(u-\Phi)^+\right]^{p(x)-1} v\,\mathrm{d}x\quad \text{for all } u,v\in L^{p(\cdot)}(\Omega).
$$

It is clear that $u\in K$ if and only if $Pu=0$, that is, P is a penalty operator of K.

For each $n \in \mathbb{N}$, we consider the following perturbed problem corresponding to problem (1.1)

$$\begin{aligned} & - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) + \beta|u|^{\theta(x)-2}u \\ & \quad + \frac{1}{\varepsilon_n}\left[(u-\Phi_n)^+\right]^{p(x)-1} \in f(x,u,\nabla u) && \text{in } \Omega, \\ & u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

The weak solutions of problem (4.2) are understood in the following sense.

Definition 4.3. *A function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is called a weak solution of problem (4.2) if there exists $\eta \in L^{r(\cdot)'}(\Omega)$ such that*

$$\eta(x) \in f(x,u(x),\nabla u(x))$$

for a. a. $x \in \Omega$ and

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right)\cdot \nabla v \,\mathrm{d}x \\ &\quad + \int_\Omega \beta|u|^{\theta(x)-2}uv\,\mathrm{d}x + \frac{1}{\varepsilon_n}\int_\Omega \left[(u-\Phi_n)^+\right]^{p(x)-1} v\,\mathrm{d}x \\ &= \int_\Omega \eta(x)v\,\mathrm{d}x \end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$.

The main result in this section about the existence and convergence properties of problem (4.2) is given as follows.

Theorem 4.4. *Let hypotheses* (H1)–(H3), (H5), *and* (H6) *be satisfied.*

(i) *For each $n \in \mathbb{N}$, the set $\mathcal{S}_n$ of the weak solutions to problem* (4.2) *is nonempty, bounded and weakly closed in $W_0^{1,\mathcal{H}}(\Omega)$.*

(ii) *It holds*

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S}.$$

(iii) *For each $u \in s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ and any sequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ with*

$$\widetilde{u}_n \in \mathcal{T}(\mathcal{S}_n,u) \quad \textit{for each } n \in \mathbb{N},$$

there exists a subsequence of $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ converging strongly to u in $W_0^{1,\mathcal{H}}(\Omega)$, where the set $\mathcal{T}(\mathcal{S}_n,u)$ is defined by

$$\mathcal{T}(\mathcal{S}_n,u) := \left\{\widetilde{u} \in \mathcal{S}_n \,:\, \|u-\widetilde{u}\| \leq \|u-v\| \textit{ for all } v \in \mathcal{S}_n\right\}.$$

Proof. (i) Let $n \in \mathbb{N}$ be fixed. Taking Lemma 4.1 into account, we see that operator P_n defined in (4.1) is demicontinuous, monotone and bounded. Similar to the proof of Theorem 3.3, consider $B(\cdot) + \frac{1}{\varepsilon_n} P_n$ instead of $B(\cdot)$, we can show that the solution set of problem (4.2) is nonempty, bounded and weakly closed.

(ii) We divide the proof of this part into three steps.

Step I: The set $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is uniformly bounded in $W_0^{1,\mathcal{H}}(\Omega)$.

Let us suppose that $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is unbounded in $W_0^{1,\mathcal{H}}(\Omega)$. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ (for a subsequence if necessary) with $u_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$ such that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty.$$

Thus, for each $n \in \mathbb{N}$, we can find $\eta_n \in L^{r(\cdot)'}(\Omega)$ with

$$\eta_n(x) \in f(x, u_n(x), \nabla u_n(x))$$

for a. a. $x \in \Omega$ such that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n + \mu(x)|\nabla u_n|^{q(x)-2}\nabla u_n\right) \cdot \nabla v \,\mathrm{d}x \\
&\quad + \int_\Omega \beta |u_n|^{\theta(x)-2} u_n v \,\mathrm{d}x + \frac{1}{\varepsilon_n}\int_\Omega \left[(u_n - \Phi_n)^+\right]^{p(x)-1} v \,\mathrm{d}x \\
&= \int_\Omega \eta_n(x) v \,\mathrm{d}x
\end{aligned} \tag{4.3}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Choosing $v = -u_n$ in (4.3), we obtain

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p(x)} + \mu(x)|\nabla u_n|^{q(x)}\right) \mathrm{d}x + \int_\Omega \beta |u_n|^{\theta(x)} \,\mathrm{d}x \\
&= \frac{1}{\varepsilon_n}\int_\Omega \left[(u_n - \Phi_n)^+\right]^{p(x)-1} (-u_n) \,\mathrm{d}x + \int_\Omega \eta_n(x) u_n \,\mathrm{d}x \\
&\le \int_\Omega \eta_n(x) u_n \,\mathrm{d}x,
\end{aligned}$$

where the last inequality is obtained by using the nonnegativity of $s \mapsto \left[(s - \Phi_n)^+\right]^{p(x)-1} s$ due to $\Phi_n(x) \ge 0$ for a. a. $x \in \Omega$. A simple calculation gives (similar to (3.12)) that

$$\left(1 - c_f - d_f \lambda_{1,p_-}^{-1}\right) \min\{\|\nabla u_n\|_{\mathcal{H}}^{q_+}, \|\nabla u_n\|_{\mathcal{H}}^{p_-}\}$$

$$
\begin{aligned}
&\le \left(1 - c_f - d_f\lambda_{1,p_-}^{-1}\right)\rho_{\mathcal{H}}(\nabla u_n)\\
&\le \|\beta_f\|_1 + M_8
\end{aligned}
$$

for some $M_8 > 0$. Passing to the limit as $n \to \infty$ for the estimates above, we get a contradiction. Therefore, the set $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$ is uniformly bounded in $W_0^{1,\mathcal{H}}(\Omega)$. This proves Step I.

Let $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ be a sequence such that $u_n \in \mathcal{S}_n$ for each $n \in \mathbb{N}$. Based on Step I, we may assume that

$$
u_n \xrightarrow{w} u \quad \text{as } n \to \infty \tag{4.4}
$$

for some $u \in W_0^{1,\mathcal{H}}(\Omega)$ and for a subsequence if necessary. Thus, the set $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is nonempty.

In the next step, we are going to show that $w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ is a subset of $\mathcal{S}$. For any fixed $u \in w\text{-}\limsup_{n\to\infty} \mathcal{S}_n$, passing to a subsequence if necessary, we are able to find a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $u_n \in \mathcal{S}_n$ for all $n \in \mathbb{N}$ such that (4.4) is satisfied. Our aim is to show that $u \in \mathcal{S}$.

Step II: $u \in K$, that is, $u(x) \le \Phi(x)$ for a. a. $x \in \Omega$.

For every $n \in \mathbb{N}$, we have $\eta_n \in \mathcal{N}_f(u_n)$ and

$$
\frac{1}{\varepsilon_n}\int_\Omega (u_n - \Phi_n)^+ v \,\mathrm{d}x = \langle Au_n + Bu_n, -v\rangle + \int_\Omega \eta_n(x) v \,\mathrm{d}x. \tag{4.5}
$$

For any $\delta > 0$, by applying Young's inequality, Hölder's inequality and hypothesis (H3)(iii), we obtain

$$
\begin{aligned}
&\int_\Omega \eta_n(x) v \,\mathrm{d}x \le \int_\Omega \left(a_f|\nabla u_n|^{\frac{p(x)(r(x)-1)}{r(x)}} + b_f|u_n|^{r(x)-1} + \alpha_f(x)\right) v \,\mathrm{d}x\\
&\le \int_\Omega \left(\delta|\nabla u_n|^{p(x)} + c_1(\delta)|v|^{r(x)} + \delta|u_n|^{r(x)} + c_2(\delta)|v|^{r(x)}\right)\mathrm{d}x \qquad (4.6)\\
&\quad + \left[\frac{1}{r_-} + \frac{1}{r'_-}\right]\|\alpha_f\|_{r'(\cdot)}\|v\|_{r(\cdot)}
\end{aligned}
$$

for some $c_1(\delta), c_2(\delta) > 0$. Using (4.6) in (4.5), by applying the boundedness of A and B (see Proposition 2.4), the convergence (4.4) and the continuity of the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$, there exists a constant $M_9 > 0$, which is independent of n, such that

$$
\frac{1}{\varepsilon_n}\int_\Omega \left[(u_n - \Phi_n)^+\right]^{p(x)-1} v \,\mathrm{d}x \le M_9(1 + \|v\|),
$$

or equivalently,

$$\int_\Omega \left[(u_n - \Phi_n)^+\right]^{p(x)-1} v \,\mathrm{d}x \le \varepsilon_n M_9 (1 + \|v\|) \tag{4.7}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Letting $n \to \infty$ in (4.7), using the convergence (4.4), the compactness of the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned}
\int_\Omega \left[(u - \Phi)^+\right]^{p(x)-1} v \,\mathrm{d}x &= \int_\Omega \lim_{n\to\infty} \left[(u_n - \Phi_n)^+\right]^{p(x)-1} v \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_\Omega \left[(u_n - \Phi_n)^+\right]^{p(x)-1} v \,\mathrm{d}x \\
&\le \lim_{n\to\infty} \varepsilon_n M_9 (1 + \|v\|) \\
&= 0
\end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. This implies $(u(x) - \Phi(x))^+ = 0$ for a. a. $x \in \Omega$ and so, $u(x) \le \Phi(x)$ for a. a. $x \in \Omega$. Hence, $u \in K$.

Step III: $u \in \mathcal{S}$. First, we know that

$$\begin{aligned}
\langle Au_n, u_n - v\rangle = \langle Bu_n, v - u_n\rangle &+ \frac{1}{\varepsilon_n} \int_\Omega \left[(u_n - \Phi_n)^+\right]^{p(x)-1} (v - u_n) \,\mathrm{d}x \\
&+ \int_\Omega \eta_n(x)(u_n - v) \,\mathrm{d}x
\end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. The monotonicity of P_n implies that

$$\begin{aligned}
\langle Au_n, u_n - v\rangle \le \langle Bu_n, v - u_n\rangle &+ \frac{1}{\varepsilon_n} \int_\Omega \left[(v - \Phi_n)^+\right]^{p(x)-1} (v - u_n) \,\mathrm{d}x \\
&+ \int_\Omega \eta_n(x)(u_n - v) \,\mathrm{d}x
\end{aligned} \tag{4.8}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$.

For any $w \in K$, we claim that there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $v_n \in K_n$ such that

$$v_n \to w \quad \text{in } W_0^{1,\mathcal{H}}(\Omega). \tag{4.9}$$

Let $w \in K$ be arbitrary, but fixed. Using hypothesis (H6), we construct a sequence $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ defined by

$$v_n = \frac{w\Phi_n}{\Phi} \quad \text{for all } n \in \mathbb{N}.$$

Since $w \in K$ and $w(x) \leq \Phi(x)$, $\Phi_n(x) \geq 0$ and $\Phi(x) > 0$ for a. a. $x \in \Omega$ by hypothesis (H6), it holds

$$v_n(x) = \frac{w(x)\Phi_n(x)}{\Phi(x)} \leq \frac{\Phi(x)\Phi_n(x)}{\Phi(x)} = \Phi_n(x) \quad \text{for a. a. } x \in \Omega.$$

Thus, $v_n \in K_n$. Applying Lebesgue's Dominated Convergence Theorem and hypothesis (H6), we have

$$\begin{aligned}
&\lim_{n\to\infty} \rho_{\mathcal{H}}(\nabla(v_n - w)) \\
&= \lim_{n\to\infty} \int_\Omega |\nabla(v_n - w)|^{p(x)} \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_\Omega \left|\nabla w\Big(1 - \frac{\Phi_n(x)}{\Phi(x)}\Big) + w\nabla\Big(\frac{\Phi_n(x)}{\Phi(x)}\Big)\right|^{p(x)} \mathrm{d}x \\
&= \int_\Omega \lim_{n\to\infty} \left|\nabla w\Big(1 - \frac{\Phi_n(x)}{\Phi(x)}\Big) + w\nabla\Big(\frac{\Phi_n(x)}{\Phi(x)}\Big)\right|^{p(x)} \mathrm{d}x \\
&= 0.
\end{aligned}$$

This combined with Proposition 2.2(v) ensures that

$$v_n \to w \quad \text{in } W_0^{1,\mathcal{H}}(\Omega).$$

Therefore, for each $w \in K$, there exists a sequence $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $v_n \in K_n$ such that (4.9) holds.

From Step II, we know that $u \in K$. So, we can find a sequence $\{w_n\}_{n\in\mathbb{N}} \subset W_0^{1,\mathcal{H}}(\Omega)$ with $w_n \in K_n$ for each $n \in \mathbb{N}$ such that $w_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$. Inserting $v = w_n$ into (4.8) and using the definition of K_n, we have

$$\langle Au_n, u_n - w_n\rangle \leq \langle Bu_n, w_n - u_n\rangle + \int_\Omega \eta_n(x)(u_n - w_n)\,\mathrm{d}x \tag{4.10}$$

with $\eta_n \in \mathcal{N}_f(u_n)$. Recall that $\mathcal{N}_f$ is bounded (see the proof of Theorem 3.3), we know that $\{\eta_n\}_{n\in\mathbb{N}}$ is bounded in $L^{r'(\cdot)}(\Omega)$. Passing to a relabeled subsequence if necessary, we may assume that

$$\eta_n \xrightarrow{w} \eta \quad \text{in } L^{r'(\cdot)}(\Omega) \quad \text{for some } \eta \in L^{r'(\cdot)}(\Omega).$$

Moreover, we use the boundedness of A and the convergence $w_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ in order to get

$$\begin{aligned}
\limsup_{n\to\infty}\langle Au_n, u_n - w_n\rangle &\geq \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle + \liminf_{n\to\infty}\langle Au_n, u - w_n\rangle \\
&= \limsup_{n\to\infty}\langle Au_n, u_n - u\rangle.
\end{aligned} \tag{4.11}$$

Since the embeddings of $W_0^{1,\mathcal{H}}(\Omega)$ into $L^{r(\cdot)}(\Omega)$ and of $W_0^{1,\mathcal{H}}(\Omega)$ into $L^{\theta(\cdot)}(\Omega)$ are compact, we get

$$\lim_{n\to\infty}\left[\langle Bu_n, w_n-u_n\rangle+\int_\Omega \eta_n(x)(u_n-w_n)\,\mathrm{d}x\right]=0. \tag{4.12}$$

Passing to the upper limit as $n\to\infty$ in (4.10) and using (4.11) as well as (4.12), it follows that

$$\limsup_{n\to\infty}\langle Au_n, u_n-w_n\rangle \le 0.$$

Since A is of type (S_+) (see Theorem 2.4), we conclude that $u_n\to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n\to\infty$. Thus, we have

$$w\text{-}\limsup_{n\to\infty}\mathcal{S}_n\subset s\text{-}\limsup_{n\to\infty}\mathcal{S}_n$$

and combined with

$$s\text{-}\limsup_{n\to\infty}\mathcal{S}_n\subset w\text{-}\limsup_{n\to\infty}\mathcal{S}_n$$

it follows that

$$\emptyset\ne w\text{-}\limsup_{n\to\infty}\mathcal{S}_n= s\text{-}\limsup_{n\to\infty}\mathcal{S}_n.$$

Arguing as in the proof of Theorem 3.3, we can prove that $\eta\in\mathcal{N}_f(u)$. For any $w\in K$, there exists a sequence $\{v_n\}_{n\in\mathbb{N}}\subset W_0^{1,\mathcal{H}}(\Omega)$ such that $v_n\in K_n$ and $v_n\to w$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n\to\infty$. Taking $v=v_n$ in (4.8) and passing to the limit as $n\to\infty$ yields

$$\begin{aligned}\langle Au, w-u\rangle+\langle Bu, w-u\rangle &= \lim_{n\to\infty}\left[\langle Au_n, v_n-u_n\rangle+\langle Bu_n, v_n-u_n\rangle\right]\\ &\ge \lim_{n\to\infty}\int_\Omega \eta_n(x)(v_n-u_n)\,\mathrm{d}x\\ &= \int_\Omega \eta(x)(w-u)\,\mathrm{d}x.\end{aligned}$$

Since $w\in K$ is arbitrary and $\eta\in\mathcal{N}_f(u)$, we infer that $u\in K$ is a solution of problem (1.1), namely, $u\in\mathcal{S}$. We conclude that

$$\emptyset\ne w\text{-}\limsup_{n\to\infty}\mathcal{S}_n= s\text{-}\limsup_{n\to\infty}\mathcal{S}_n\subset\mathcal{S}.$$

(iii) For any fixed

$$u\in s\text{-}\limsup_{n\to\infty}\mathcal{S}_n,$$

the nonemptiness, boundedness and closedness of $\mathcal{S}_n$ guarantees that the set $\mathcal{T}(\mathcal{S}_n, u)$ is well-defined. Let $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ be any sequence such that

$$\widetilde{u}_n \in \mathcal{T}(\mathcal{S}_n, u) \quad \text{ for each } n \in \mathbb{N}.$$

From Step I, we know that the sequence $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ is bounded. Hence, we may suppose that

$$\widetilde{u}_n \xrightarrow{w} \widetilde{u} \quad \text{in } W_0^{1,\mathcal{H}}(\Omega) \tag{4.13}$$

for some $\widetilde{u} \in W_0^{1,\mathcal{H}}(\Omega)$. Similar to the argument in Step II, we get that $\widetilde{u} \in K$. Therefore, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle A\widetilde{u}_n, \widetilde{u}_n - v\rangle &= \langle B\widetilde{u}_n, v - \widetilde{u}_n\rangle + \frac{1}{\varepsilon_n}\int_\Omega \left[(\widetilde{u}_n - \Phi_n)^+\right]^{p(x)-1}(v - \widetilde{u}_n)\,\mathrm{d}x\\ &\quad + \int_\Omega \eta_n(x)(\widetilde{u}_n - v)\,\mathrm{d}x\end{aligned}$$

for all $v \in W_0^{1,\mathcal{H}}(\Omega)$. Arguing exactly as in the proof of Step III, we derive that $\widetilde{u}$ is a solution of problem (1.1). Because of $u \in s\text{-}\limsup_{n\to\infty}\mathcal{S}_n$, passing to a subsequence if necessary, we can find a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \in \mathcal{S}_n$ and $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$ as $n \to \infty$. This fact along with (4.13) gives

$$\|\widetilde{u} - u\| \le \liminf_{n\to\infty}\|\widetilde{u}_n - u\| \le \liminf_{n\to\infty}\|u_n - u\| = 0.$$

Hence, $\widetilde{u} = u$. This finishes the proof of the theorem. $\square$

In the last part of this section, we consider some special cases of Theorem 4.4.

If $\Phi_n = \Phi$ for each $n \in \mathbb{N}$, we have the following result.

Corollary 4.5. *Let hypotheses* (H1)–(H5) *be satisfied.*

(i) *For each $n \in \mathbb{N}$, the set $\mathcal{S}_n$ of weak solutions of the problem*

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) + \beta|u|^{\theta(x)-2}u\\ &+\frac{1}{\varepsilon_n}\left[(u-\Phi)^+\right]^{p(x)-1} \in f(x,u,\nabla u) && \text{in } \Omega,\\ &u = 0 && \text{on } \partial\Omega,\end{aligned}$$

is nonempty, bounded and weakly closed in $W_0^{1,\mathcal{H}}(\Omega)$.

(ii) *It holds*

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \mathcal{S}_n = s\text{-}\limsup_{n\to\infty} \mathcal{S}_n \subset \mathcal{S},$$

where $\mathcal{S}$ *is the solution set to problem* (1.1).

(iii) *For each* $u \in s\text{-}\limsup_{n\to\infty} \mathcal{S}_n$ *and any sequence* $\{u_n\}_{n\in\mathbb{N}}$ *with*

$$u_n \in \mathcal{T}(\mathcal{S}_n, u) \quad \text{for each } n \in \mathbb{N},$$

there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ *converging strongly to* u *in* $W_0^{1,\mathcal{H}}(\Omega)$, *where the set* $\mathcal{T}(\mathcal{S}_n, u)$ *is defined by*

$$\mathcal{T}(\mathcal{S}_n, u) := \{u \in \mathcal{S}_n \,:\, \|u-u\| \leq \|u-v\| \text{ for all } v \in \mathcal{S}_n\}.$$

If $\beta = 0$, then Theorem 4.4 reduces the following corollary.

Corollary 4.6. *Let hypotheses* (H1), (H3), (H5), *and* (H6) *be satisfied.*

(i) *For each* $n \in \mathbb{N}$, *the set* $\tilde{\mathcal{S}_n}$ *of weak solutions of the problem*

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \mu(x)|\nabla u|^{q(x)-2}\nabla u\right) & \\ +\frac{1}{\varepsilon_n}(u-\Phi_n)^+ \in f(x,u,\nabla u) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

is nonempty, bounded and weakly closed in $W_0^{1,\mathcal{H}}(\Omega)$.

(ii) *It holds*

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \tilde{\mathcal{S}_n} = s\text{-}\limsup_{n\to\infty} \tilde{\mathcal{S}_n} \subset \tilde{\mathcal{S}},$$

where $\tilde{\mathcal{S}}$ *is the solution set to problem* (1.2).

(iii) *For each* $u \in s\text{-}\limsup_{n\to\infty} \tilde{\mathcal{S}_n}$ *and any sequence* $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ *with*

$$\tilde{u}_n \in \mathcal{T}(\tilde{\mathcal{S}_n}, u) \quad \text{for each } n \in \mathbb{N},$$

there exists a subsequence of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ *converging strongly to* u *in* $W_0^{1,\mathcal{H}}(\Omega)$, *where the set* $\mathcal{T}(\tilde{\mathcal{S}_n}, u)$ *is defined by*

$$\mathcal{T}(\tilde{\mathcal{S}_n}, u) := \left\{\tilde{u} \in \tilde{\mathcal{S}_n} \,:\, \|u-\tilde{u}\| \leq \|u-v\| \text{ for all } v \in \tilde{\mathcal{S}_n}\right\}.$$

If p, q, θ are constants and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$ such that $1 < p < N$, $p < q < p^*$ and $\theta < p^*$, then Theorem 4.4 becomes the following.

Corollary 4.7. *Let hypotheses* (H3), (H5), *and* (H6) *be satisfied.*

(i) *For each* $n \in \mathbb{N}$, *the set* $\tilde{\mathcal{S}}_n$ *of weak solutions of the problem*

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + \beta|u|^{\theta-2}u & \\ +\frac{1}{\varepsilon_n}(u-\Phi_n)^+ \in f(x,u,\nabla u) & \quad \text{in } \Omega, \\ u = 0 & \quad \text{on } \partial\Omega, \end{aligned}$$

is nonempty, bounded and weakly closed in $W_0^{1,\mathcal{H}}(\Omega)$.

(ii) *It holds*

$$\emptyset \neq w\text{-}\limsup_{n\to\infty} \tilde{\mathcal{S}}_n = s\text{-}\limsup_{n\to\infty} \tilde{\mathcal{S}}_n \subset \tilde{\mathcal{S}},$$

where $\tilde{\mathcal{S}}$ *is the solution set to problem* (1.5).

(iii) *For each* $u \in s\text{-}\limsup_{n\to\infty} \tilde{\mathcal{S}}_n$ *and any sequence* $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ *with*

$$\tilde{u}_n \in \mathcal{T}(\tilde{\mathcal{S}}_n, u) \quad \textit{for each } n \in \mathbb{N},$$

there exists a subsequence of $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ *converging strongly to* u *in* $W_0^{1,\mathcal{H}}(\Omega)$, *where the set* $\mathcal{T}(\tilde{\mathcal{S}}_n, u)$ *is defined by*

$$\mathcal{T}(\tilde{\mathcal{S}}_n, u) := \left\{\tilde{u} \in \tilde{\mathcal{S}}_n \,:\, \|u-\tilde{u}\| \le \|u-v\| \textit{ for all } v \in \tilde{\mathcal{S}}_n\right\}.$$

More particularly, if p, q, θ are constants and $0 \le \mu(\cdot) \in L^\infty(\Omega)$ such that $1 < p < N$, $p < q < p^*$, $\theta < p^*$, $\beta = 0$ and $\Phi_n = \Phi$, then Corollary 4.7 coincides with Theorem 3.4 of Zeng, Bai, Gasiński and Winkert [47].

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