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Positive solutions for Neumann problems with indefinite and unbounded potential

Nikolaos S. Papageorgiou[a], Vicenţiu D. Rădulescu[b,*]

[a] National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

[b] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

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ABSTRACT

We consider semilinear Neumann equations with an indefinite and unbounded potential. We establish the existence and uniqueness of positive solutions. We show that our setting incorporates as special cases several parametric equations of interest (such as the equidiffusive logistic equation).

1. Introduction

In this paper we deal with the following semilinear Neumann problem

$$-\Delta u(z)+\beta(z)u(z)=f(z,u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega. \tag{1}$$

Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary, β is a potential function which is in general unbounded and sign changing. The reaction $f(z,x)$ is a Carathéodory function (that is, for all $x\in\mathbb{R}$ the mapping $z\longmapsto f(z,x)$ is measurable and for a.a. $z\in\Omega$, $x\longmapsto f(z,x)$ is continuous), which exhibits general growth conditions near $+\infty$ and near 0^+. As we will see these conditions incorporate in our framework as special cases various parametric problems such as equidiffusive logistic equations. We are interested in the existence and uniqueness of positive solutions.

Recently semilinear Neumann problems with unbounded and indefinite potential were studied by Papageorgiou and Rădulescu [1]. They deal with equations in which the reaction $f(z,x)$ exhibits an asymmetric behavior at $+\infty$ and at $-\infty$ (jumping nonlinearity) and they prove multiplicity theorems providing sign information for all the solutions. We mention also the recent works of Mugnai and Papageorgiou [2], who examine equations driven by the p-Laplacian plus an indefinite potential and of Papageorgiou and Smyrlis [3], who consider a special class of coercive semilinear equations.

2. Positive solutions

In the analysis of problem (1) we will use the Sobolev space $H^1(\Omega)$ and the ordered Banach space $C^1(\overline{\Omega})$. The positive cone of the latter space is given by

$$C_+=\{u\in C^1(\overline{\Omega}) : u(z)\geqslant 0 \text{ for all } z\in\overline{\Omega}\}.$$

* Corresponding author.
E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), vicentiu.radulescu@imar.ro, vicentiu.radulescu@math.cnrs.fr (V.D. Rădulescu).

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This cone has a nonempty interior

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

The hypotheses on the potential function $\beta(\cdot)$ are the following.

$H(\beta)$: $\beta \in L^s(\Omega)$ with $s > \frac{N}{2}$ and $\beta^+ \in L^\infty(\Omega)$

By $\{\hat{\lambda}_n\}_{n \geqslant 1}$ we denote the distinct eigenvalues of the differential operator $u \to -\Delta u + \beta(z)u$, $u \in H^1(\Omega)$. We know that $\hat{\lambda}_1$ is simple and

$$\hat{\lambda}_1 = \inf\left[\frac{\xi(u)}{\|u\|_2^2} : u \in H^1(\Omega),\ u \neq 0\right] \tag{2}$$

where $\xi(u) = \|Du\|_2^2 + \int_\Omega \beta(z)u^2 dz$ for all $u \in H^1(\Omega)$. The infimum in (2) is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1$. From (2) it is clear that the elements of this eigenspace do not change sign. By $\hat{u}_1$ we denote the positive L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. The regularity results of Wang [4] imply that $\hat{u}_1 \in C_+ \backslash \{0\}$ and in fact using $H(\beta)$ and the maximum principle of Vazquez [5], we have $\hat{u}_1 \in \operatorname{int} C_+$. The next lemma is a consequence of these properties (see Papageorgiou and Rădulescu [1]).

Lemma 1. *If $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \leqslant \hat{\lambda}_1$ a.e. in Ω, $\vartheta \neq \hat{\lambda}_1$, then there exists $c_0 > 0$ such that*

$$\xi(u) - \int_\Omega \vartheta(z)u^2 dz \geqslant c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega).$$

The hypotheses on the reaction $f(z,x)$ are as follows.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$ with $a \in L^\infty(\Omega)_+$, $2 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } 3 \leqslant N \\ +\infty & \text{if } N = 1, 2 \end{cases}$;

(ii) $\limsup_{x \to +\infty} \frac{f(z,x)}{x} \leqslant \vartheta(z)$ uniformly for a.a. $z \in \Omega$, with $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \hat{\lambda}_1$ a.e in Ω, $\vartheta \not\equiv \hat{\lambda}_1$;

(iii) $\liminf_{x \to 0^+} \frac{f(z,x)}{x} \geqslant \eta(z)$ uniformly for a.a. $z \in \Omega$, with $\eta \in L^\infty(\Omega)$, $\eta(z) \geqslant \hat{\lambda}_1$ a.e. in Ω, $\eta \not\equiv \hat{\lambda}_1$.

Remark 1. Since we are interested in positive solutions and the above hypotheses concern the positive semi-axis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. Note that according to hypothesis $H(f)$ (i), $f(z,\cdot)$ has subcritical growth. Finally hypotheses $H(f)$ (ii), (iii) imply that the quotient $\frac{f(z,x)}{x}$ crosses at least the principal eigenvalue $\hat{\lambda}_1$ as we move from 0^+ to $+\infty$.

From the spectral analysis of Papageorgiou and Rădulescu [1] we know that there exists $\gamma_0 > \max\{-\hat{\lambda}_1, 1\}$ such that

$$\xi(u) + \gamma_0\|u\|_2^2 \geqslant c_1\|u\|^2 \quad \text{for all } u \in H^1(\Omega),\ \text{some } c_1 > 0. \tag{3}$$

We introduce the Carathéodory function $\hat{f}(z,x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ f(z,x) + \gamma_0 x & \text{if } 0 < x \end{cases}$ and its primitive $\hat{F}(z,x) = \int_0^x \hat{f}(z,s)ds$.

Proposition 2. *If hypotheses $H(\beta)$ and $H(f)$ hold, then problem (1) has at least one positive solution $u_0 \in \operatorname{int} C_+$.*

Proof. Let $\hat{\varphi} : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\hat{\varphi}(u) = \frac{1}{2}\xi(u) + \frac{\gamma_0}{2}\|u\|_2^2 - \int_\Omega \hat{F}(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Hypotheses $H(f)$ (i), (ii) imply that given $\epsilon > 0$, we can find $c_2 = c_2(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(\vartheta(z) + \epsilon)(x^+)^2 + c_2 \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R} \tag{4}$$

(recall that for all $x \in \mathbb{R}$, $x^\pm = \max\{\pm x, 0\}$). Then

$$\hat{\varphi}(u) \geqslant \frac{1}{2}\left[\xi(u^+) - \int_\Omega \vartheta(z)(u^+)^2 dz\right] - \frac{\epsilon}{2}\|u^+\|^2 + \frac{1}{2}\xi(u^-) + \frac{\gamma_0}{2}\|u^-\|_2^2 - c_2|\Omega|_N \quad \text{(see (4))},$$

where $u^\pm(\cdot) = u(\cdot)^\pm$ and $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. Using Lemma 1 and (3), we obtain

$$\hat{\varphi}(u) \geqslant \frac{c_0 - \epsilon}{2}\|u^+\|^2 + \frac{c_1}{2}\|u^-\|^2 - c_2|\Omega|_N.$$

Choosing $\epsilon \in (0, c_0)$, we see that $\hat{\varphi}$ is coercive. Also, using the Sobolev embedding theorem, we check that $\hat{\varphi}$ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in H^1(\Omega)$ such that

$$\hat{\varphi}(u_0) = \inf[\hat{\varphi}(u) : u \in H^1(\Omega)], \tag{5}$$

Hypothesis $H(f)_1$ (iii) implies that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$F(z, x) \geqslant \frac{1}{2}(\eta(z) - \epsilon)x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0, \delta]. \tag{6}$$

Let $t \in (0, 1)$ be small such that $t\hat{u}_1(z) \in (0, \delta]$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_1 \in \operatorname{int} C_+$). Then

$$\hat{\varphi}(t\hat{u}_1) = \frac{t^2}{2}\xi(\hat{u}_1) - \int_\Omega F(z, t\hat{u}_1)dz \leqslant \frac{t^2}{2}\int_\Omega (\hat{\lambda}_1 - \eta(z))\hat{u}_1^2 dz + \frac{t^2\epsilon}{2} \quad \text{(see (6) and recall that } \|\hat{u}_1\|_2 = 1).$$

Note that $\hat{\mu} = \int_\Omega (\eta(z) - \hat{\lambda}_1)\hat{u}_1^2 dz > 0$ (see hypothesis $H(f)$ (iii)). Hence

$$\hat{\varphi}(t\hat{u}_1) \leqslant \frac{t^2}{2}[-\hat{\mu} + \epsilon].$$

So, choosing $\epsilon \in (0, \hat{\mu})$, we see that $\hat{\varphi}(t\hat{u}_1) < 0$. Therefore

$$\hat{\varphi}(u_0) < 0 = \hat{\varphi}(0) \quad \text{(see (5)); hence } u_0 \neq 0.$$

From (5) we have $\hat{\varphi}'(u_0) = 0$ and so

$$\int_\Omega (Du_0, Dh)_{\mathbb{R}^N} dz + \int_\Omega \beta(z)u_0 h dz + \gamma_0 \int_\Omega u_0 h dz = \int_\Omega \hat{f}(z, u_0)h dz \quad \text{for all } h \in H^1(\Omega). \tag{7}$$

In (7) we choose $h = -u_0^- \in H^1(\Omega)$. Then

$$\xi(u_0^-) + \gamma_0\|u_0^-\|_2^2 = 0; \quad \text{hence } u_0 \geqslant 0, \ u_0 \neq 0 \text{ (see (3))}.$$

Then from (7) and Green's identity we obtain

$$-\Delta u_0(z) + \beta(z)u_0(z) = f(z, u_0(z)) \quad \text{a.e. in } \Omega, \qquad \frac{\partial u_0}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

From Wang [4] we obtain $u_0 \in C_+ \backslash \{0\}$. Note that hypotheses $H(f)$ (i), (iii) imply that given $\rho > 0$, we can find $\xi_\rho > 0$ such that $f(z, x) + \xi_\rho x \geqslant 0$ for a.a. $z \in \Omega$, all $x \in [0, \rho]$. If $\rho = \|u_0\|_\infty$ and $\xi_\rho > 0$ as above, then

$$\begin{aligned} &-\Delta u_0(z) + (\beta(z) + \xi_\rho)u_0(z) \geqslant 0 \quad \text{a.e. in } \Omega, \\ &\Rightarrow \Delta u_0(z) \leqslant (\|\beta^+\|_\infty + \xi_\rho)u_0(z) \quad \text{a.e. in } \Omega, \\ &\Rightarrow u_0 \in \operatorname{int} C_+ \quad \text{(see Vazquez [5])}. \quad \Box \end{aligned}$$

If we strengthen the conditions on $f(z, \cdot)$ we can assure uniqueness of the positive solution. The new stronger hypotheses on the reaction $f(z, x)$ are as follows.

$H(f)' : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, and hypotheses $H(f)'$ (i)–(iii) are the same as the corresponding hypotheses $H(f)$ (i)–(iii)

(iv) for a.a. $z \in \Omega$, $x \to \frac{f(z,x)}{x}$ is decreasing and for all $z \in \Omega_0 \subseteq \Omega$ with $|\Omega_0|_N > 0$, $x \to \frac{f(z,x)}{x}$ is strictly decreasing on $(0, +\infty)$.

Proposition 3. *If hypotheses $H(\beta)$ and $H(f)'$ hold, then problem (1) admits a unique positive solution $u_0 \in \operatorname{int} C_+$.*

Proof. From Proposition 2, we already have one solution $u_0 \in \operatorname{int} C_+$. Let y be another positive solution of (1). As before we can show that $y \in \operatorname{int} C_+$.

We consider the integral functional $\tau : L^2(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\tau(u) = \begin{cases} \frac{1}{2}\|Du^{1/2}\|_2^2 + \frac{1}{2}\int_\Omega \beta(z)u dz & \text{if } u \geqslant 0, \ u^{1/2} \in H^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

From Lemma 1 of Diaz and Saa [6], we know that $\tau(\cdot)$ is convex and lower semicontinuous. Since $u_0, y \in \operatorname{int} C_+$, we see that given any $h \in C^1(\overline{\Omega})$ for $|\lambda| \leqslant 1$ small we have

$$u_0^2 + \lambda h, \ y^2 + \lambda h \in dom\tau = \{u \in H^1(\Omega) : \tau(u) < +\infty\} \quad \text{(effective domain of } \tau).$$

Therefore τ is Gâteaux differentiable at u_0^2 and at y^2 in direction h. Moreover, using the chain rule, we have

$$\begin{aligned} \tau'(u_0^2)(h) &= \frac{1}{2}\int_\Omega \frac{-\Delta u_0}{u_0} h dz + \frac{1}{2}\int_\Omega \beta(z)h dz \\ \tau'(y^2)(h) &= \frac{1}{2}\int_\Omega \frac{-\Delta y}{y} h dz + \frac{1}{2}\int_\Omega \beta(z)h dz \quad \text{for all } h \in H^1(\Omega) \end{aligned}$$

(recall that $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$). The convexity of τ implies the monotonicity of τ'. Therefore

$$0 \leqslant \frac{1}{2}\int_\Omega \left(\frac{f(z,u_0)}{u_0} - \frac{f(z,y)}{y}\right)(u_0^2 - y^2)dz \leqslant 0,$$
$$\Rightarrow u_0 = y \quad \text{(see hypothesis } H(f)' \text{ (iv) and recall that } u_0, y \in \operatorname{int} C_+). \quad \square$$

3. Special cases

First consider the following logistic equation with equidiffusive reaction

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z) - h(z, u(z)) \quad \text{in}\,\Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{8}$$

The hypotheses on the perturbation $h(z, x)$ are as follows.

$H(h)_1$: $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $h(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|h(z, x)| \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)$ and $2 < r < 2^*$;
(ii) $\lim_{x\to+\infty} \frac{h(z,x)}{x} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iii) $\lim_{x\to0^+} \frac{h(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;
(iv) for a.a. $z \in \Omega$, $x \to \frac{h(z,x)}{x}$ is increasing and for all $z \in \Omega_0 \subseteq \Omega$ with $|\Omega_0|_N > 0$, $x \to \frac{h(z,x)}{x}$ is strictly increasing.

Remark 2. A typical perturbation $h(z, x)$ satisfying hypotheses $H(h)_1$ is $h(z, x) = h(x) = x^{r-1}$ with $2 < r < 2^*$, which corresponds to the classical equidiffusive logistic equation.

Proposition 4. *If hypotheses $H(\beta)$ and $H(f)_1$ hold and $\lambda > \hat{\lambda}_1$, then problem (8) admits a unique positive solution $u_0 \in \operatorname{int} C_+$.*

Now consider the following nonhomogeneous eigenvalue problem:

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z)^{q-1} \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{9}$$

Proposition 5. *If hypotheses $H(\beta)$ hold, $q \in (2, 2^*)$ and $\lambda > 0$, then problem (9) has a unique positive solution $u_0 \in \operatorname{int} C_+$.*

Finally consider the following parametric Neumann problem:

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z) + h(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{10}$$

The hypotheses on the perturbation $h(z, x)$ are as follows.

$H(h)_2$: $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $h(z, 0) = 0$, $h(z, x) > 0$ for all $x > 0$ and

(i) $h(z, x) \leqslant a(z)(1 + x^{p-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$ with $a \in L^\infty(\Omega)_+$;
(ii) $\lim_{x\to+\infty} \frac{h(z,x)}{x} = 0$ uniformly for a.a. $z \in \Omega$;
(iii) $\lim_{x\to0^+} \frac{h(z,x)}{x} = +\infty$ uniformly for a.a. $z \in \Omega$;
(iv) for a.a. $z \in \Omega$, $x \to \frac{h(z,x)}{x}$ is decreasing and for all $z \in \Omega_0 \subseteq \Omega$ with $|\Omega_0|_N > 0$, $x \to \frac{h(z,x)}{x}$ is strictly decreasing on $(0, +\infty)$.

Proposition 6. *If hypotheses $H(\beta)$ and $H(h)_2$ hold and $\lambda < \hat{\lambda}_1$, then problem (10) admits a unique positive solution $u_0 \in \operatorname{int} C_+$.*

In fact we can show that this upper bound $\hat{\lambda}_1$ is sharp.

Proposition 7. *If hypotheses $H(\beta)$ and $H(h)_2$ hold and $\lambda \geqslant \hat{\lambda}_1$, then problem (10) has no positive solution.*

Proof. Let $\lambda \geqslant \hat{\lambda}_1$ and suppose that problem (10) has a positive solution u_λ. As in the proof of Proposition 2, we can show that $u_\lambda \in \operatorname{int} C_+$. Let

$$R(\hat{u}_1, u_\lambda)(z) = \left|D\hat{u}_1(z)\right|^2 - |Du_\lambda(z)|^{p-2}\left(Du_\lambda(z), D\left(\frac{\hat{u}_1^2}{u_\lambda}\right)\right)_{\mathbb{R}^N}.$$

From Picone's identity (see, for example, Gasinski and Papageorgiou [7, p. 785]), we have

$$\begin{aligned} 0 \leqslant \int_\Omega R(\hat{u}_1, u_\lambda)dz &= \|D\hat{u}_1\|_2^2 - \int_\Omega (-\Delta u_\lambda)\frac{\hat{u}_1^2}{u_\lambda}dz \quad \text{(by Green's identity)} \\ &= \|D\hat{u}_1\|_2^2 - \int_\Omega (\lambda - \beta(z))\hat{u}_1^2 dz - \int_\Omega h(z, u_\lambda)\frac{\hat{u}_1^2}{u_\lambda}dz \\ &< (\hat{\lambda}_1 - \lambda) \leqslant 0 \end{aligned}$$

a contradiction. $\square$

Remark 3. Analogous results can be obtained for nonlinear equations driven by the p-Laplacian.

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